

Lecture 2: Positive definite matrices and eigenvalues; the OLS criterion and normal equations

Econometrics ■ Unit I: Introduction and generalized least squares

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This lecture

Previous lecture

Lecture 1: Preliminaries: notation; differentiation with respect to a vector; expectations and variances.

Today

- Positive definite matrices and eigenvalues
- Review of ordinary least squares

A symmetric matrix A is

- *positive definite* if $a' A a > 0$ for every nonzero vector a ;
- *positive semidefinite* if $a' A a \geq 0$ for every a .

If X has full column rank k , then $X'X$ is positive definite, because for $a \neq 0$

$$a' X' X a = (Xa)'(Xa) = \sum_i (Xa)_i^2 > 0,$$

since $Xa \neq 0$ when the columns of X are linearly independent.

A symmetric $k \times k$ matrix A can be written as

$$A = V\Lambda V', \quad V'V = VV' = I, \quad \Lambda = \text{diag}(\lambda_1, \dots, \lambda_k),$$

where $\lambda_1, \dots, \lambda_k$ are its eigenvalues and the columns of V are the corresponding eigenvectors. Then

$$\begin{aligned} \text{tr}(A) &= \sum_j \lambda_j, & |A| &= \prod_j \lambda_j, \\ A^{-1} &= V\Lambda^{-1}V' \quad (\text{if all } \lambda_j \neq 0). \end{aligned}$$

For a positive definite matrix all $\lambda_j > 0$.

Assume $E(\epsilon) = 0$, $\text{Var}(\epsilon) = \sigma^2 I_n$ and $\text{rank}(X) = k$. OLS chooses b to minimise the residual sum of squares

$$S(b) = (y - Xb)'(y - Xb).$$

Step 1 (expand). Since $b'X'y$ is a scalar, it equals its transpose $y'Xb$. Hence

$$\begin{aligned} S(b) &= y'y - y'Xb - b'X'y + b'X'Xb \\ &= y'y - 2b'X'y + b'X'Xb. \end{aligned}$$

Step 2 (differentiate). By Rules P1 and P2,

$$\frac{\partial S}{\partial b} = -2X'y + 2X'Xb.$$

Step 3 (equate to zero). $X'Xb = X'y$. These are the *normal equations*. Since $X'X$ is invertible,

$$b = (X'X)^{-1}X'y.$$

Step 4 (check minimum). The matrix of second derivatives is $2X'X$, which is positive definite. So b minimises $S(b)$.

Mean. Substitute $y = X\beta + \varepsilon$:

$$b = (X'X)^{-1}X'(X\beta + \varepsilon) = \beta + (X'X)^{-1}X'\varepsilon.$$

By Rule P4, $E(b) = \beta + (X'X)^{-1}X'E(\varepsilon) = \beta$. So b is unbiased.

Practice

Re-work the derivations and examples of this lecture without looking at the slides.

Next lecture

Lecture 3: Variance of OLS; estimating σ^2 ; the Gauss–Markov theorem; a numerical example.