

Lecture 3: Variance of OLS; estimating σ^2 ; the Gauss–Markov theorem; a numerical example

Econometrics ■ Unit I: Introduction and generalized least squares

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Previous lecture

Lecture 2: Positive definite matrices and eigenvalues; the OLS criterion and normal equations.

Today

- Review of ordinary least squares
- A numerical example

Variance. From $b - \beta = (X'X)^{-1}X'\epsilon$ and Rule P4,

$$\begin{aligned}\text{Var}(b) &= (X'X)^{-1}X'(\sigma^2 I)X(X'X)^{-1} \\ &= \sigma^2(X'X)^{-1}X'X(X'X)^{-1} = \sigma^2(X'X)^{-1}.\end{aligned}$$

Estimating σ^2 . The residual vector is $e = y - Xb$. Put $H = X(X'X)^{-1}X'$ and $M = I - H$. Then

$$e = y - Hy = My = M(X\beta + \epsilon) = MX\beta + M\epsilon = M\epsilon,$$

because $MX = X - X(X'X)^{-1}X'X = X - X = 0$.

Also $M' = M$ and

$$M^2 = (I - H)(I - H) = I - 2H + H^2 = I - 2H + H = M,$$

since $H^2 = X(X'X)^{-1}X'X(X'X)^{-1}X' = H$.

Therefore $e'e = \varepsilon' M' M \varepsilon = \varepsilon' M \varepsilon$, and by Rule P6,

$$E(e'e) = \sigma^2 \text{tr}(M) = \sigma^2 \{\text{tr}(I_n) - \text{tr}(H)\} = \sigma^2(n - k),$$

using $\text{tr}(H) = \text{tr}\{(X'X)^{-1}X'X\} = \text{tr}(I_k) = k$ (Rule P5).

Hence

$$s^2 = \frac{e'e}{n - k}$$

is an unbiased estimator of σ^2 .

Gauss–Markov theorem (OLS)

Under these assumptions, b is the best linear unbiased estimator (BLUE) of β .

A self-contained proof is the special case $\Omega = I$ of Proof II of Aitken's theorem in Chapter 2.

A numerical example

For $x = (0, 1, 2)'$ and $y = (1, 2, 4)'$, fit $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$. Here

$$X = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad X'X = \begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix}, \quad X'y = \begin{pmatrix} 7 \\ 10 \end{pmatrix}.$$

Now $|X'X| = 15 - 9 = 6$ and

$$b = \frac{1}{6} \begin{pmatrix} 5 & -3 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} 7 \\ 10 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 35 - 30 \\ -21 + 30 \end{pmatrix} = \begin{pmatrix} 5/6 \\ 3/2 \end{pmatrix}.$$

The fitted values are $(5/6, 7/3, 23/6)'$, the residuals are $e = (1/6, -1/3, 1/6)'$, and $e'e = \frac{1}{36} + \frac{4}{36} + \frac{1}{36} = \frac{1}{6}$. Since $n - k = 1$, $s^2 = 1/6$.

Practice

Re-work the numerical example of OLS without looking at the slides.

Next lecture

Lecture 4: Meaning and nature of econometrics; exact and econometric relationships.