

Lecture 8: The model with a general covariance matrix; what happens to OLS

Econometrics ■ Unit I: Introduction and generalized least squares

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This lecture

Previous lecture

Lecture 7: From a question to an estimated model; role of econometrics.

Today

- The model with a general covariance matrix
- What happens to OLS?

The generalized linear regression model

OLS assumes $\text{Var}(\epsilon) = \sigma^2 I_n$: all disturbances have the same variance and are uncorrelated. Such disturbances are called *spherical*.

In many economic data sets this fails. We therefore consider the *generalized linear regression model*

$$y = X\beta + \epsilon, \quad E(\epsilon) = 0, \quad \text{Var}(\epsilon) = \sigma^2 \Omega, \quad (2.1)$$

where

- Ω is a known $n \times n$ symmetric positive definite matrix,
- $\sigma^2 > 0$ may be unknown, and
- $\text{rank}(X) = k < n$.

- **Heteroskedasticity** (Chapter 4): unequal variances but no correlation,

$$\Omega = \text{diag}(\omega_1, \dots, \omega_n).$$

- **Autocorrelation** (Chapter 5): disturbances correlated over time. For example, for three periods with first-order autocorrelation,

$$\Omega = \frac{1}{1 - \rho^2} \begin{pmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{pmatrix}.$$

The diagonal entries of $\sigma^2\Omega$ are the variances of the disturbances. The off-diagonal entries are their covariances.

OLS is still unbiased. As in the Preliminaries,

$$b = (X'X)^{-1}X'y = \beta + (X'X)^{-1}X'\varepsilon,$$

so $E(b) = \beta + (X'X)^{-1}X'E(\varepsilon) = \beta$. This step used only $E(\varepsilon) = 0$, not the form of $\text{Var}(\varepsilon)$.

The variance of OLS changes. By Rule P4,

$$\begin{aligned}\text{Var}(b) &= (X'X)^{-1}X' \text{Var}(\varepsilon) X(X'X)^{-1} \\ &= \sigma^2(X'X)^{-1}X'\Omega X(X'X)^{-1}.\end{aligned}$$

This equals the usual formula $\sigma^2(X'X)^{-1}$ only when $X'\Omega X = X'X$, which is true when $\Omega = I$. So the usual standard errors printed by OLS are generally wrong.

s^2 is biased; OLS is no longer best

s^2 **is biased**. Using $e = M\varepsilon$ and Rule P6,

$$E(e'e) = E(\varepsilon' M \varepsilon) = \sigma^2 \text{tr}(M\Omega),$$

so

$$E(s^2) = \frac{\sigma^2 \text{tr}(M\Omega)}{n - k},$$

which is not σ^2 in general.

OLS is no longer best. We shall see that another linear unbiased estimator, GLS, has smaller variance.

Example 2.1 (Averaging correlated observations)

Let $y_i = \mu + \varepsilon_i$ ($i = 1, \dots, n$). OLS gives $\bar{y}$. Then

$$\begin{aligned}\text{Var}(\bar{y}) &= \frac{1}{n^2} \text{Var}\left(\sum_i \varepsilon_i\right) \\ &= \frac{1}{n^2} \left\{ \sum_i \text{Var}(\varepsilon_i) + 2 \sum_{i < j} \text{Cov}(\varepsilon_i, \varepsilon_j) \right\}.\end{aligned}$$

- If all variances are σ^2 and all covariances are zero, this is σ^2/n .
- If the covariances are positive, the second term is positive and $\text{Var}(\bar{y})$ is larger than σ^2/n . The usual formula σ^2/n would then understate the uncertainty.

Practice

- Exercise 2.1 (Exercises and Solutions booklet).

Next lecture

Lecture 9: The GLS estimator by transforming the model.