

Lecture 9: The GLS estimator by transforming the model

Econometrics ■ Unit I: Introduction and generalized least squares

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This lecture

Previous lecture

Lecture 8: The model with a general covariance matrix; what happens to OLS.

Today

- The GLS estimator by transforming the model

The idea; Step 1: a matrix T

The idea is simple: transform the model so that the new disturbances are spherical, and then apply OLS.

Step 1: A matrix T with $T\Omega T' = I$. Since Ω is symmetric positive definite, write

$$\Omega = Q\Lambda Q', \quad Q'Q = QQ' = I, \quad \Lambda = \text{diag}(\lambda_1, \dots, \lambda_n),$$

all $\lambda_i > 0$. Define

$$T = \Lambda^{-1/2}Q', \quad \Lambda^{-1/2} = \text{diag}(\lambda_1^{-1/2}, \dots, \lambda_n^{-1/2}).$$

Step 1 (cont.): $T\Omega T' = I_n$ and $T'T = \Omega^{-1}$

Then

$$\begin{aligned} T\Omega T' &= \Lambda^{-1/2} Q' (Q\Lambda Q') Q \Lambda^{-1/2} \\ &= \Lambda^{-1/2} (Q'Q) \Lambda (Q'Q) \Lambda^{-1/2} \\ &= \Lambda^{-1/2} \Lambda \Lambda^{-1/2} = I_n, \end{aligned}$$

and

$$T'T = Q\Lambda^{-1/2} \Lambda^{-1/2} Q' = Q\Lambda^{-1} Q' = \Omega^{-1}.$$

T is nonsingular because all $\lambda_i > 0$.

Step 2: Transform the model. Premultiply (2.1) by T :

$$Ty = TX\beta + T\epsilon, \quad \text{that is,} \quad y^* = X^*\beta + \epsilon^*,$$

where $y^* = Ty$, $X^* = TX$ and $\epsilon^* = T\epsilon$. The coefficient vector β is unchanged.

Step 3: The new disturbances are spherical. By Rule P4,

$$\begin{aligned} E(\epsilon^*) &= T E(\epsilon) = 0, \\ \text{Var}(\epsilon^*) &= T \text{Var}(\epsilon) T' = \sigma^2 T \Omega T' = \sigma^2 I_n. \end{aligned}$$

Also $\text{rank}(X^*) = \text{rank}(TX) = k$ because T is nonsingular.

Step 4: apply OLS to the transformed model

$$\begin{aligned}\hat{\beta}_G &= (X^{*'}X^*)^{-1}X^{*'}y^* \\ &= \{(TX)'(TX)\}^{-1}(TX)'(Ty) \\ &= (X'T'TX)^{-1}X'T'Ty \\ &= (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y.\end{aligned}$$

Definition 2.1

The estimator

$$\boxed{\hat{\beta}_G = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y} \quad (2.2)$$

is called the **generalized least squares (GLS) estimator** or the **Aitken estimator** of β .

If $\Omega = I$, then (2.2) becomes $(X'X)^{-1}X'y = b$. So OLS is the special case of GLS with spherical disturbances.

Remember

$$\hat{\beta}_G = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y, \quad \text{Var}(\hat{\beta}_G) = \sigma^2(X'\Omega^{-1}X)^{-1}.$$

Practice

Re-work the derivations and examples of this lecture without looking at the slides.

Next lecture

Lecture 10: The GLS estimator by direct minimisation.