Lecture 10: The GLS estimator by direct minimisation

Econometrics ▪ Unit I: Introduction and generalized least squares

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This lecture

Previous lecture

Lecture 9: The GLS estimator by transforming the model.

Today

- The GLS estimator by direct minimisation

Minimising a weighted sum of squares

The same estimator is obtained by minimising the *weighted* sum of squares

$$Q(b) = (y - Xb)'\Omega^{-1}(y - Xb).$$

This is the OLS criterion for the transformed model, because

$$\begin{aligned}(y^* - X^*b)'(y^* - X^*b) &= (y - Xb)'T'T(y - Xb) \\ &= (y - Xb)'\Omega^{-1}(y - Xb).\end{aligned}$$

Steps 1 to 3: the GLS normal equations

Step 1 (expand). Since Ω^{-1} is symmetric, $b'X'\Omega^{-1}y = y'\Omega^{-1}Xb$ (both are the same scalar). Hence

$$Q(b) = y'\Omega^{-1}y - 2b'X'\Omega^{-1}y + b'X'\Omega^{-1}Xb.$$

Step 2 (differentiate). By Rule P1 with $a = X'\Omega^{-1}y$ and Rule P2 with $A = X'\Omega^{-1}X$,

$$\frac{\partial Q}{\partial b} = -2X'\Omega^{-1}y + 2X'\Omega^{-1}Xb.$$

Step 3 (equate to zero). We obtain the GLS normal equations

$$X'\Omega^{-1}X\,b = X'\Omega^{-1}y.$$

Steps 4 and 5: solution and minimum

Step 4 (solve). $X'\Omega^{-1}X$ is invertible because, for $a \neq 0$,

$$a'X'\Omega^{-1}Xa = (Xa)'\Omega^{-1}(Xa) > 0,$$

since $Xa \neq 0$ and Ω^{-1} is positive definite. So $b = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y = \widehat{\beta}_G$.

Step 5 (minimum). The second-derivative matrix is $2X'\Omega^{-1}X$, which is positive definite. Hence $\widehat{\beta}_G$ minimises $Q(b)$.

Remark

Multiplying Ω by a positive constant c does not change $\widehat{\beta}_G$, because c cancels:

$$\{X'(c\Omega)^{-1}X\}^{-1}X'(c\Omega)^{-1}y = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y.$$

Only the *relative* sizes of variances and covariances matter.

After this lecture

Practice

- Exercises 2.2 and 2.4 (Exercises and Solutions booklet).

Next lecture

Lecture 11: Mean and variance of the Aitken estimator.