

# Lecture 11: Mean and variance of the Aitken estimator

Econometrics ■ Unit I: Introduction and generalized least squares

Kapil Kumar

Department of Statistics  
Chaudhary Charan Singh University, Meerut

B.Sc. (Hons) Statistics, Semester V  
Academic Session 2026–2027

## Previous lecture

Lecture 10: The GLS estimator by direct minimisation.

## Today

- Mean and variance of the Aitken estimator

**Unbiasedness.** Substitute  $y = X\beta + \varepsilon$  into (2.2):

$$\begin{aligned}\hat{\beta}_G &= (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}(X\beta + \varepsilon) \\ &= (X'\Omega^{-1}X)^{-1}(X'\Omega^{-1}X)\beta + (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}\varepsilon \\ &= \beta + (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}\varepsilon.\end{aligned}$$

Taking expectations,  $E(\hat{\beta}_G) = \beta$ .

**Variance.** Write  $A_G = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}$ , so  $\hat{\beta}_G - \beta = A_G\varepsilon$ . By Rule P4,

$$\begin{aligned}\text{Var}(\hat{\beta}_G) &= A_G(\sigma^2\Omega)A'_G \\ &= \sigma^2(X'\Omega^{-1}X)^{-1}X'\Omega^{-1}\Omega\Omega^{-1}X(X'\Omega^{-1}X)^{-1} \\ &= \sigma^2(X'\Omega^{-1}X)^{-1}(X'\Omega^{-1}X)(X'\Omega^{-1}X)^{-1} \\ &= \sigma^2(X'\Omega^{-1}X)^{-1}.\end{aligned}$$

Let  $e_G = y - X\hat{\beta}_G$  be the GLS residuals. In the transformed model the residuals are  $e_G^* = Te_G$  and

$$e_G^{*'}e_G^* = e_G'T'Te_G = e_G'\Omega^{-1}e_G.$$

The transformed model satisfies all OLS assumptions, so by the Preliminaries  $E(e_G^{*'}e_G^*) = \sigma^2(n - k)$ . Therefore

$$s_G^2 = \frac{e_G'\Omega^{-1}e_G}{n - k}$$

is an unbiased estimator of  $\sigma^2$ , and  $\widehat{\text{Var}}(\hat{\beta}_G) = s_G^2(X'\Omega^{-1}X)^{-1}$ .

### Practice

Re-work the derivations and examples of this lecture without looking at the slides.

### Next lecture

Lecture 12: Aitken's theorem: statement and short proof.