

Lecture 12: Aitken's theorem: statement and short proof

Econometrics ■ Unit I: Introduction and generalized least squares

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B.Sc. (Hons) Statistics, Semester V
Academic Session 2026–2027

Previous lecture

Lecture 11: Mean and variance of the Aitken estimator.

Today

- Gauss–Markov theorem for GLSE (Aitken's theorem)
 - Proof I (short): through the transformed model

Theorem 2.2 (Aitken's theorem; Aitken, 1935)

In model (2.1), the GLS estimator $\hat{\beta}_G$ is the best linear unbiased estimator (BLUE) of β . That is, if $\tilde{\beta}$ is any estimator that is linear in y and satisfies $E(\tilde{\beta}) = \beta$ for every β , then $\text{Var}(\tilde{\beta}) - \text{Var}(\hat{\beta}_G)$ is positive semidefinite.

Two proofs are given.

- Proof I is short and uses the ordinary Gauss–Markov theorem (for OLS) as a known result.
- Proof II is longer, does not use any earlier theorem, and works with the most general linear estimator $Ay + c$.

Proof I (short): through the transformed model

Proof.

Step 1. By Section 2.3 of the notes, with $T\Omega T' = I$, the transformed model $y^* = X^*\beta + \varepsilon^*$ ($y^* = Ty$, $X^* = TX$, $\varepsilon^* = T\varepsilon$) satisfies $E(\varepsilon^*) = 0$, $\text{Var}(\varepsilon^*) = \sigma^2 I$ and $\text{rank}(X^*) = k$. It satisfies all the assumptions of the ordinary Gauss–Markov theorem.

Step 2. Let $\tilde{\beta} = Ay$ be linear and unbiased. (A constant term c can be ignored, because unbiasedness forces $c = 0$; see Step 1 of Proof II.) Since T is nonsingular, $y = T^{-1}y^*$, so

$$\tilde{\beta} = AT^{-1}y^*,$$

which is linear in y^* . It is unbiased for β in the transformed model, because it is the same estimator.

Proof (cont.)

Step 3. By the ordinary Gauss–Markov theorem applied to the transformed model, the OLS estimator $(X^{*'}X^*)^{-1}X^{*'}y^*$ has the smallest variance among all linear unbiased estimators of β .

Step 4. By Section 2.3 of the notes, this OLS estimator equals $(X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y = \hat{\beta}_G$. Hence $\text{Var}(\tilde{\beta}) - \text{Var}(\hat{\beta}_G)$ is positive semidefinite, and $\hat{\beta}_G$ is BLUE. □

Idea in one line: GLS is just OLS on a model that satisfies the classical assumptions, so it gets the BLUE property of OLS.

Practice

- Exercise 2.6 (Exercises and Solutions booklet).

Next lecture

Lecture 13: Aitken's theorem: the direct proof; GLS versus OLS.