Lecture 14: A worked example of GLS

Econometrics ▪ Unit I: Introduction and generalized least squares

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This lecture

Previous lecture

Lecture 13: Aitken's theorem: the direct proof; GLS versus OLS.

Today

- A worked example

A worked example

Example 2.2 (Estimating a common mean with correlated errors)

Let $y_i = \mu + \varepsilon_i$, $i = 1, 2, 3$, with

$$y = \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix}, \qquad \Omega = \begin{pmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The first two errors are positively correlated; the third is uncorrelated with them. Here $X = (1, 1, 1)'$.

A worked example (cont.)

Example 2.2 (cont.)

Step 1: Ω^{-1}. The upper 2×2 block has determinant $1 - 1/4 = 3/4$, so its inverse is

$$\frac{4}{3}\begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix} = \begin{pmatrix} 4/3 & -2/3 \\ -2/3 & 4/3 \end{pmatrix}.$$

Hence

$$\Omega^{-1} = \begin{pmatrix} 4/3 & -2/3 & 0 \\ -2/3 & 4/3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Example 2.2 (cont.)

Step 2: the two products. The row sums of Ω^{-1} are $2/3,\ 2/3,\ 1$. Therefore

$$X'\Omega^{-1}X = \tfrac{2}{3} + \tfrac{2}{3} + 1 = \tfrac{7}{3},$$
$$X'\Omega^{-1}y = \tfrac{2}{3}(2) + \tfrac{2}{3}(4) + 1(8) = \tfrac{4}{3} + \tfrac{8}{3} + 8 = 12.$$

Step 3: estimates.

$$\widehat{\mu}_G = \frac{12}{7/3} = \frac{36}{7} \approx 5.1429, \qquad \widehat{\mu}_{OLS} = \bar{y} = \frac{14}{3} \approx 4.6667.$$

Equivalently $\widehat{\mu}_G = (2y_1 + 2y_2 + 3y_3)/7$: each of the correlated observations gets weight $2/7$ and the independent one gets $3/7$.

Example 2.2 (cont.)

Step 4: variances.

$$\mathrm{Var}(\widehat{\mu}_G) = \sigma^2 (X'\Omega^{-1}X)^{-1} = \tfrac{3}{7}\sigma^2.$$

For OLS, using the example in Section 2.2 of the notes with variances $\sigma^2, \sigma^2, \sigma^2$ and one covariance $\sigma^2/2$,

$$\mathrm{Var}(\bar{y}) = \frac{1}{9}\{3\sigma^2 + 2(\sigma^2/2)\} = \frac{4}{9}\sigma^2.$$

Since $\frac{4}{9} - \frac{3}{7} = \frac{28-27}{63} = \frac{1}{63} > 0$, GLS is more efficient, as Aitken's theorem says.

Example 2.2 (cont.)

Step 5: estimate of σ^2. The residuals are

$$e_G = (2 - \tfrac{36}{7},\ 4 - \tfrac{36}{7},\ 8 - \tfrac{36}{7})' = (-\tfrac{22}{7},\ -\tfrac{8}{7},\ \tfrac{20}{7})'.$$

Then

$$\begin{aligned} e_G' \Omega^{-1} e_G &= \frac{1}{49}\left\{\tfrac{4}{3}(484) + \tfrac{4}{3}(64) - 2 \cdot \tfrac{2}{3}(22)(8) + 400\right\} \\ &= \frac{1}{49}\{496 + 400\} = \frac{128}{7}, \end{aligned}$$

so

$$s_G^2 = \frac{128/7}{3-1} = \frac{64}{7} \qquad \text{and} \qquad \widehat{\mathrm{Var}}(\hat{\mu}_G) = \frac{64}{7} \cdot \frac{3}{7} = \frac{192}{49}.$$

After this lecture

Practice

- Exercises 2.3 and 2.5 (Exercises and Solutions booklet).

Next lecture

Lecture 15: Feasible GLS; summary of key formulas.