

Lecture 20: Diagnostics: condition indices and variance-decomposition proportions

Econometrics ■ Unit II: Multicollinearity and heteroskedasticity

Kapil Kumar

Department of Statistics
Chaudhary Charan Singh University, Meerut

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Previous lecture

Lecture 19: Diagnostics: correlation matrix, $|R|$, partial regressions and VIFs.

Today

- Tools for handling the problem: diagnostics
 - Condition number and condition indices
 - Eigenvalues and variance-decomposition proportions

Let $\lambda_1, \dots, \lambda_k$ be the eigenvalues of $X'X$ (standardised data), with largest $\lambda_{\max}$ and smallest $\lambda_{\min}$. Small eigenvalues indicate near-linear dependencies.

Definition 3.4

The **condition number** is $\kappa = \lambda_{\max}/\lambda_{\min}$. The **condition indices** are $\kappa_j = \lambda_{\max}/\lambda_j$, $j = 1, \dots, k$.

The largest condition index equals the condition number. The usual guidelines (Montgomery et al., 2021) are:

- $\kappa < 100$: no serious (non-harmful) multicollinearity;
- $100 \leq \kappa \leq 1000$: moderate to strong multicollinearity (the “danger level”);
- $\kappa > 1000$: severe multicollinearity.

- The number of large condition indices equals the number of near-linear dependencies.
- A limitation is that κ and κ_j are unbounded ($1 \leq \kappa < \infty$).

Singular values. If $X = UDV'$ is the singular value decomposition with singular values $\mu_1, \dots, \mu_k$, then $X'X = VD^2V'$, so $\lambda_j = \mu_j^2$.

- Some books define condition indices as $\eta_j = \mu_{\max}/\mu_j = \sqrt{\kappa_j}$, with correspondingly smaller cut-offs.
- Always state which definition is used.

This diagnostic, due to Belsley et al. (1980), shows which coefficients are affected by each near-dependency.

Step 1. Write $X'X = V\Lambda V'$, where $V = (v_{ij})$ is the orthogonal matrix of eigenvectors and $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_k)$. Then $(X'X)^{-1} = V\Lambda^{-1}V'$.

Step 2. The i th diagonal element of $V\Lambda^{-1}V'$ is $\sum_j v_{ij} \cdot \frac{1}{\lambda_j} \cdot v_{ij}$. Hence

$$\text{Var}(b_i) = \sigma^2 \sum_{j=1}^k \frac{v_{ij}^2}{\lambda_j}, \quad \text{VIF}_i = \sum_{j=1}^k \frac{v_{ij}^2}{\lambda_j}.$$

So the variance of b_i is split into k parts, one for each eigenvalue. A small λ_j makes its part large.

Step 3. The **proportion of the variance of b_i due to the j th eigenvalue** is

$$p_{ij} = \frac{v_{ij}^2 / \lambda_j}{\text{VIF}_i}, \quad \sum_{j=1}^k p_{ij} = 1.$$

Procedure.

1. Compute the condition indices $\kappa_1, \dots, \kappa_k$.
2. Identify the eigenvalues whose condition index is above the danger level. Their number is the number of near-dependencies.
3. For each such eigenvalue λ_j , look at the proportions $p_{1j}, \dots, p_{kj}$.
4. If $p_{ij} > 0.5$ for two or more coefficients, those coefficients are adversely affected by that dependency, and the corresponding variables are involved in it.

This diagnostic is more informative than the others: it shows the presence, the number and the location of the dependencies.

Example 3.3 (Diagnostics for two highly correlated regressors)

Let $X'X = \begin{pmatrix} 1 & 0.95 \\ 0.95 & 1 \end{pmatrix}$.

Eigenvalues. $|X'X - \lambda I| = (1 - \lambda)^2 - 0.95^2 = 0$ gives $1 - \lambda = \pm 0.95$, so $\lambda_1 = 1.95$ and $\lambda_2 = 0.05$.

Eigenvectors. For $\lambda_1 = 1.95$: $(1 - 1.95)v_1 + 0.95v_2 = 0 \Rightarrow v_1 = v_2$, so $v = (1, 1)'/\sqrt{2}$. For $\lambda_2 = 0.05$: $0.95v_1 + 0.95v_2 = 0 \Rightarrow v_2 = -v_1$, so $v = (1, -1)'/\sqrt{2}$. All squared entries v_{ij}^2 equal $1/2$.

VIF. $VIF_1 = VIF_2 = 1/(1 - 0.95^2) = 1/0.0975 = 10.256$. Check with Step 2:

$$VIF_1 = \frac{0.5}{1.95} + \frac{0.5}{0.05} = 0.2564 + 10 = 10.2564.$$

Example 3.3 (cont.)

Proportions. $p_{12} = 10/10.2564 = 0.975$ and $p_{11} = 0.025$; the same holds for b_2 . Both coefficients have 97.5% of their variance from the small eigenvalue, so both are involved in the same dependency.

Condition number. $\kappa = 1.95/0.05 = 39$, **determinant**
 $D = 1 - 0.95^2 = 0.0975$.

Here VIF exceeds 10 while κ is below 100. Rules of thumb from different diagnostics need not agree, which is why several diagnostics should be examined together.

Practice

- Exercise 3.4 (Exercises and Solutions booklet).

Next lecture

Lecture 21: Remedies: more data, dropping variables, prior information, generalized inverse, principal components.