

Lecture 32: Moments of the AR(1) disturbances

Econometrics ■ Unit III: Autocorrelation and instrumental variables

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Previous lecture

Lecture 31: Autocorrelation: concept, measures and causes; the AR(1) scheme.

Today

- The first-order autoregressive scheme
 - Moments of the AR(1) disturbances

Step 1 (successive substitution).

$$\begin{aligned}u_t &= v_t + \rho u_{t-1} \\&= v_t + \rho(v_{t-1} + \rho u_{t-2}) = v_t + \rho v_{t-1} + \rho^2 u_{t-2} \\&= v_t + \rho v_{t-1} + \rho^2 v_{t-2} + \rho^3 u_{t-3} = \cdots = \sum_{j=0}^{\infty} \rho^j v_{t-j}.\end{aligned}$$

Since $|\rho| < 1$, $\rho^j \rightarrow 0$ and the effect of old shocks dies out.

Step 2 (mean).

$$E(u_t) = \sum_j \rho^j E(v_{t-j}) = 0.$$

Step 3 (variance). The v 's are independent, so

$$\text{Var}(u_t) = \sum_{j=0}^{\infty} \rho^{2j} \sigma_v^2 = \sigma_v^2 (1 + \rho^2 + \rho^4 + \cdots) = \frac{\sigma_v^2}{1 - \rho^2},$$

using the geometric series $1 + r + r^2 + \cdots = 1/(1 - r)$ with $r = \rho^2 < 1$. Call this σ_u^2 . It does not depend on t .

Step 4 (covariance at lag 1). Multiply (5.1) by u_{t-1} and take expectations:

$$\text{E}(u_t u_{t-1}) = \rho \text{E}(u_{t-1}^2) + \text{E}(v_t u_{t-1}) = \rho \sigma_u^2 + 0,$$

since v_t is independent of u_{t-1} . So $\text{Cov}(u_t, u_{t-1}) = \rho \sigma_u^2$.

Step 5 (lag s). Multiply (5.1) by u_{t-s} ($s \geq 1$) and take expectations:

$$\gamma_s = E(u_t u_{t-s}) = \rho E(u_{t-1} u_{t-s}) + E(v_t u_{t-s}) = \rho \gamma_{s-1}.$$

Repeating,

$$\gamma_s = \rho \gamma_{s-1} = \rho^2 \gamma_{s-2} = \cdots = \rho^s \gamma_0 = \rho^s \sigma_u^2.$$

Hence

$$\text{Cov}(u_t, u_{t-s}) = \rho^s \sigma_u^2, \quad \rho_s = \rho^s.$$

So ρ is the correlation between consecutive disturbances, and correlations decline geometrically with the lag.

Step 6 (covariance matrix). Collecting these results for

$$u = (u_1, \dots, u_n)',$$

$$\text{Var}(u) = \sigma_u^2 \begin{pmatrix} 1 & \rho & \rho^2 & \dots & \rho^{n-1} \\ \rho & 1 & \rho & \dots & \rho^{n-2} \\ \vdots & & \ddots & & \vdots \\ \rho^{n-1} & \rho^{n-2} & \dots & \rho & 1 \end{pmatrix} = \sigma_v^2 \Omega,$$

where

$$\Omega = \frac{1}{1 - \rho^2} (\rho^{|t-s|}).$$

This is the GLS model of Chapter 2.

Example 5.1

If $\rho = 0.6$ and $\sigma_v^2 = 4$, then $\sigma_u^2 = 4/(1 - 0.36) = 6.25$.

- The covariance of consecutive disturbances is $0.6(6.25) = 3.75$; two periods apart it is $0.36(6.25) = 2.25$.
- The correlations are 0.6 and 0.36.

Remember

$$\text{AR}(1): \text{Var}(u_t) = \frac{\sigma_v^2}{1 - \rho^2}, \quad \text{Cov}(u_t, u_{t-s}) = \rho^s \sigma_u^2.$$

Practice

- Exercises 5.1 and 5.2 (Exercises and Solutions booklet).

Next lecture

Lecture 33: Consequences of autocorrelated disturbances.