

Lecture 37: ρ known: the generalized difference equation and the Prais–Winsten transformation

Econometrics ■ Unit III: Autocorrelation and instrumental variables

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This lecture

Previous lecture

Lecture 36: Durbin's h test.

Today

- Solution of autocorrelation
 - When ρ is known: GLS by the generalized difference

If autocorrelation is caused by an omitted variable or a wrong functional form, the model itself should first be corrected. If the autocorrelation is genuine, the following methods are used.

Take $y_t = \beta_0 + \beta_1 x_t + u_t$ with AR(1) disturbances.

Step 1 (lag the equation). $y_{t-1} = \beta_0 + \beta_1 x_{t-1} + u_{t-1}$.

Step 2 (multiply by ρ and subtract).

$$\begin{aligned} y_t - \rho y_{t-1} &= (\beta_0 - \rho\beta_0) + \beta_1(x_t - \rho x_{t-1}) + (u_t - \rho u_{t-1}) \\ &= \beta_0(1 - \rho) + \beta_1(x_t - \rho x_{t-1}) + v_t. \end{aligned}$$

Step 3 (new disturbances are classical). The new disturbance v_t has constant variance σ_v^2 and is uncorrelated over time. So OLS can be applied to the *generalized difference equation*

$$y_t^* = \beta_0^* + \beta_1 x_t^* + v_t,$$
$$y_t^* = y_t - \rho y_{t-1}, \quad x_t^* = x_t - \rho x_{t-1}, \quad \beta_0^* = \beta_0(1 - \rho),$$

for $t = 2, \dots, n$, and $\beta_0 = \beta_0^*/(1 - \rho)$.

Step 4 (the first observation). The difference equation loses the first observation. To keep it (the Prais–Winsten transformation; Prais and Winsten, 1954), use

$$y_1^* = \sqrt{1 - \rho^2} y_1, \quad x_1^* = \sqrt{1 - \rho^2} x_1, \quad \text{constant column } \sqrt{1 - \rho^2}.$$

Its disturbance $\sqrt{1 - \rho^2} u_1$ has variance

$$(1 - \rho^2) \sigma_u^2 = (1 - \rho^2) \cdot \frac{\sigma_v^2}{1 - \rho^2} = \sigma_v^2,$$

the same as the others, and it is uncorrelated with $v_2, v_3, \dots$ because each v_t is independent of past disturbances.

The transformation in matrix form

In matrix form, this is the GLS transformation Ty , TX with

$$T = \begin{pmatrix} \sqrt{1-\rho^2} & 0 & 0 & \cdots & 0 \\ -\rho & 1 & 0 & \cdots & 0 \\ 0 & -\rho & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -\rho & 1 \end{pmatrix}, \quad \text{Var}(Tu) = \sigma_v^2 I.$$

When the full transformation is used, the constant column becomes $\sqrt{1-\rho^2}$ in the first row and $1-\rho$ in the other rows, and its coefficient is β_0 itself.

Example 5.5 (Transforming the data)

Let $\rho = 0.5$, $y = (2, 4, 5)'$ and $x = (1, 2, 3)'$. Then $\sqrt{1 - \rho^2} = \sqrt{0.75} = \sqrt{3}/2$, and

t	y_t^*	constant column	x_t^*
1	$\frac{\sqrt{3}}{2}(2) = \sqrt{3}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}(1) = \frac{\sqrt{3}}{2}$
2	$4 - 0.5(2) = 3$	$1 - 0.5 = 0.5$	$2 - 0.5(1) = 1.5$
3	$5 - 0.5(4) = 3$	0.5	$3 - 0.5(2) = 2$

Example 5.5 (cont.)

With X^* formed by the last two columns,

$$X^{*'}X^* = \begin{pmatrix} \frac{3}{4} + \frac{1}{4} + \frac{1}{4} & \frac{3}{4} + 0.75 + 1 \\ \frac{3}{4} + 0.75 + 1 & \frac{3}{4} + 2.25 + 4 \end{pmatrix} = \begin{pmatrix} 5/4 & 5/2 \\ 5/2 & 7 \end{pmatrix},$$

$$X^{*'}y^* = \begin{pmatrix} \frac{3}{2} + 1.5 + 1.5 \\ \frac{3}{2} + 4.5 + 6 \end{pmatrix} = \begin{pmatrix} 9/2 \\ 12 \end{pmatrix}.$$

The determinant is $\frac{35}{4} - \frac{25}{4} = \frac{5}{2}$, so

$$\begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} = \frac{2}{5} \begin{pmatrix} 7 & -5/2 \\ -5/2 & 5/4 \end{pmatrix} \begin{pmatrix} 9/2 \\ 12 \end{pmatrix} = \frac{2}{5} \begin{pmatrix} 31.5 - 30 \\ -11.25 + 15 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 1.5 \end{pmatrix}.$$

The GLS fitted equation is $\hat{y}_t = 0.6 + 1.5x_t$.

Practice

Re-work the derivations and examples of this lecture without looking at the slides.

Next lecture

Lecture 38: ρ unknown: estimating ρ ; the Cochrane–Orcutt procedure.