Lecture 40: A complete worked example; summary of key formulas

Econometrics ▪ Unit III: Autocorrelation and instrumental variables

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This lecture

Previous lecture

Lecture 39: Durbin's two-step method; the Hildreth–Lu method.

Today

- Solution of autocorrelation
  - A complete worked example
- Summary of key formulas

A complete worked example

Example 5.6 (Detecting and correcting autocorrelation)

The following $n = 10$ observations are taken over time.

t	1	2	3	4	5	6	7	8	9	10
x_t	2	3	5	6	8	9	11	12	14	15
y_t	9.6	11.1	13.8	13.6	14.1	14.3	17.1	19.8	22.7	24.0

Step 1 (OLS). $\sum x = 85$, $\sum y = 160.1$, $\sum x^2 = 905$, $\sum xy = 1548.1$. So

$$S_{xx} = 905 - 85^2/10 = 182.5,$$
$$S_{xy} = 1548.1 - 85(160.1)/10 = 187.25.$$

Example 5.6 (cont.)

Step 1 (OLS, cont.) Hence

$$b_1 = \frac{187.25}{182.5} = 1.0260, \qquad b_0 = 16.01 - 1.0260(8.5) = 7.2888.$$

The residuals $e_t = y_t - 7.2888 - 1.0260x_t$ are

$$0.259,\ 0.733,\ 1.381,\ 0.155,\ -1.397,$$
$$-2.223,\ -1.475,\ 0.199,\ 1.047,\ 1.321.$$

They show runs of the same sign: first positive, then negative, then positive again.

A complete worked example (cont.)

Example 5.6 (cont.)

Step 2 (Durbin–Watson test). $\sum e_t^2 = 14.485$ and $\sum_{t=2}^{10}(e_t - e_{t-1})^2 = 9.394$, so

$$d = \frac{9.394}{14.485} = 0.649.$$

For $n = 10$, $k' = 1$ at the 5% level, the table gives $d_L = 0.879$ and $d_U = 1.320$. Since $0.649 < 0.879$, we reject H_0: there is positive autocorrelation.

Step 3 (estimates of ρ).

- From d: $\widehat{\rho} = 1 - 0.649/2 = 0.676.$
- Theil–Nagar: $\widehat{\rho} = \dfrac{100(0.676) + 4}{100 - 4} = 0.746.$
- From residuals: $\widehat{\rho} = \dfrac{\sum e_t e_{t-1}}{\sum e_{t-1}^2} = \dfrac{8.882}{12.741} = 0.697.$

Example 5.6 (cont.)

Step 4 (Cochrane–Orcutt). With $\widehat{\rho} = 0.697$, the regression of $y_t - 0.697 y_{t-1}$ on $x_t - 0.697 x_{t-1}$ gives $\widehat{\beta}_1 = 1.222$ and $\widehat{\beta}_0 = 5.078$. New residuals give $\widehat{\rho} = 0.721$, then 0.729, $0.733, \ldots$. The iterations settle at $\widehat{\rho} = 0.736$, with $\widehat{\beta}_0 = 4.594$ and $\widehat{\beta}_1 = 1.252$.

Step 5 (Hildreth–Lu). RSS of the transformed regression for a coarse grid:

ρ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
RSS	14.38	12.71	11.27	10.06	9.10	8.37	7.88	**7.64**	7.69	8.19

The smallest RSS is near $\rho = 0.7$. A finer grid in steps of 0.02 between 0.60 and 0.84 gives the smallest RSS (7.621) at $\rho = 0.74$, with $\widehat{\beta}_0 = 4.530$ and $\widehat{\beta}_1 = 1.256$.

Example 5.6 (cont.)

Step 6 (Durbin's two-step). The regression of y_t on 1, y_{t-1}, x_t, x_{t-1} gives coefficient 0.762 for y_{t-1}, so $\widehat{\rho} = 0.762$. The second stage then gives $\widehat{\beta}_0 = 4.180$ and $\widehat{\beta}_1 = 1.276$.

Summary.

Method	$\widehat{\rho}$	$\widehat{\beta}_0$	$\widehat{\beta}_1$
OLS (ignores autocorrelation)	–	7.289	1.026
Cochrane–Orcutt (iterated)	0.736	4.594	1.252
Hildreth–Lu	0.74	4.530	1.256
Durbin's two-step	0.762	4.180	1.276

The three correction methods agree closely with each other, and all differ clearly from OLS. With only 10 observations, small differences between the methods are expected.

Summary of key formulas

- AR(1): $u_t = \rho u_{t-1} + v_t$, $\operatorname{Var}(u_t) = \sigma_v^2/(1-\rho^2)$, $\operatorname{Cov}(u_t, u_{t-s}) = \rho^s \sigma_u^2$.
- OLS stays unbiased; usual variances understated when $\rho > 0$; OLS not BLUE.
- Durbin–Watson: $d = \sum_{t=2}^{n}(e_t - e_{t-1})^2 / \sum e_t^2 \approx 2(1-\widehat{\rho})$, $0 \le d \le 4$; compare with d_L, d_U.
- Durbin's $h = \widehat{\rho}\sqrt{n/(1 - n\widehat{\operatorname{Var}}(\widehat{\gamma}))} \sim N(0,1)$ when y_{t-1} is a regressor.
- Estimates of ρ: $1 - d/2$; Theil–Nagar $\{n^2(1-d/2) + k^2\}/(n^2 - k^2)$; $\sum e_t e_{t-1} / \sum e_{t-1}^2$.
- Generalized difference: $y_t - \rho y_{t-1} = \beta_0(1-\rho) + \beta_1(x_t - \rho x_{t-1}) + v_t$; first observation multiplied by $\sqrt{1-\rho^2}$ (Prais–Winsten).
- Unknown ρ: Cochrane–Orcutt (iterate), Durbin's two-step (coefficient of y_{t-1}), Hildreth–Lu (grid search on RSS), first differences when $\rho \approx 1$.

After this lecture

Practice

- R practical P10: Correcting autocorrelation and comparing inference.

Next lecture

Lecture 41: Why OLS fails: consistency; a regressor correlated with the disturbance.