Lecture 58: LIML: the least variance ratio and its minimisation

Econometrics ▪ Unit IV: Simultaneous equations models

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This lecture

Previous lecture

Lecture 57: 2SLS standard errors; an over-identified example; 2SLS, IV and ILS under exact identification.

Today

- Limited information maximum likelihood
  - The idea: the least variance ratio
  - Minimising the variance ratio

Limited information maximum likelihood

- LIML is due to Anderson and Rubin (1949).
- It estimates one structural equation using only the information about that equation (its included and excluded variables), not the full specification of the other equations; hence "limited information".
- Under normal disturbances, it is the maximum likelihood estimator based on this limited information.
- We derive it by the equivalent *least variance ratio* principle, which needs only calculus.

The least variance ratio

Let $H = [y_1 \; Y_1]$ (the $m+1$ endogenous variables in the equation) and $\boldsymbol{a} = (1, -\boldsymbol{\beta}')'$. For a trial value of $\boldsymbol{\beta}$, define

$$y^* = y_1 - Y_1\boldsymbol{\beta} = H\boldsymbol{a}.$$

- If $\boldsymbol{\beta}$ is the true value, then $y^* = X_1\boldsymbol{\gamma} + u$: y^* depends only on the *included* exogenous variables.
- The excluded variables X_2 should then add nothing to explaining y^*.

Two regressions of y^*

- Regress y^* on X_1 only. Its residual sum of squares is (Preliminaries, $e'e = y'My$)

$$\text{RSS}_1 = y^{*\prime} M_1 y^* = \boldsymbol{a}' H' M_1 H \boldsymbol{a} = \boldsymbol{a}' W_1 \boldsymbol{a},$$

  where $M_1 = I - X_1(X_1'X_1)^{-1}X_1'$ and $W_1 = H'M_1H$.
- Regress y^* on all exogenous variables X. Its residual sum of squares is

$$\text{RSS} = \boldsymbol{a}' W \boldsymbol{a},$$

  where $M = I - X(X'X)^{-1}X'$ and $W = H'MH$.

The variance ratio

Since X contains X_1, adding regressors cannot increase the residual sum of squares, so $\mathrm{RSS}_1 \geq \mathrm{RSS}$. Define the **variance ratio**

$$\kappa(\boldsymbol{a}) = \frac{\boldsymbol{a}' W_1 \boldsymbol{a}}{\boldsymbol{a}' W \boldsymbol{a}} \geq 1.$$

If the excluded variables have no explanatory power for y^*, κ is close to 1.

LIML chooses β to make κ as small as possible.

Minimising the variance ratio

Step 1 (differentiate). W_1 and W are symmetric, so by Rule P2 and the quotient rule,

$$\frac{\partial \kappa}{\partial \boldsymbol{a}} = \frac{2W_1\boldsymbol{a}\,(\boldsymbol{a}'W\boldsymbol{a}) - (\boldsymbol{a}'W_1\boldsymbol{a})\,2W\boldsymbol{a}}{(\boldsymbol{a}'W\boldsymbol{a})^2}.$$

Step 2 (equate to zero). Dividing by $2/(\boldsymbol{a}'W\boldsymbol{a})$,

$$W_1\boldsymbol{a} - \frac{\boldsymbol{a}'W_1\boldsymbol{a}}{\boldsymbol{a}'W\boldsymbol{a}}W\boldsymbol{a} = 0 \implies (W_1 - \kappa W)\boldsymbol{a} = 0.$$

Minimising the variance ratio (cont.)

Step 3 (determinantal equation). A nonzero $\boldsymbol{a}$ satisfies this only if

$$|W_1 - \kappa W| = 0.$$

This is a polynomial equation of degree $m+1$ in κ. At each root, κ equals the value of the ratio at the corresponding $\boldsymbol{a}$.

Step 4 (choose the smallest root). Since we want the minimum of the ratio, take the **smallest root** $\widehat{\kappa}$.

Minimising the variance ratio (cont.)

Step 5 (coefficients). Solve $(W_1 - \widehat{\kappa} W)\boldsymbol{a} = 0$, scale the solution so that its first element is 1, and read off $\widehat{\boldsymbol{\beta}}_{LIML}$ as minus the remaining elements. Then estimate $\boldsymbol{\gamma}$ by regressing $y_1 - Y_1 \widehat{\boldsymbol{\beta}}_{LIML}$ on X_1:

$$\widehat{\boldsymbol{\gamma}}_{LIML} = (X_1' X_1)^{-1} X_1' (y_1 - Y_1 \widehat{\boldsymbol{\beta}}_{LIML}).$$

The matrices W_1 and W are only $(m+1) \times (m+1)$, whatever the sample size.

After this lecture

Practice

Re-work the derivations and examples of this lecture without looking at the slides.

Next lecture

Lecture 59: LIML under exact identification; the k-class; a LIML calculation.