

Lecture 59: LIML under exact identification; the k -class; a LIML calculation

Econometrics ■ Unit IV: Simultaneous equations models

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Previous lecture

Lecture 58: LIML: the least variance ratio and its minimisation.

Today

- Limited information maximum likelihood
 - Exact identification
 - Relation with OLS and 2SLS (k -class)

- If the equation is exactly identified, X_2 has m columns. Regress $y^* = y_1 - Y_1\beta$ on X . The coefficients of X_2 in this regression are linear in β .
- There are m such coefficients and m unknowns in β . So β can be chosen to make all of them exactly zero; this choice is the ILS estimate.
- Then adding X_2 does not reduce the residual sum of squares, so $RSS_1 = RSS$ and $\kappa = 1$, the smallest possible value.
- Hence $\hat{\kappa} = 1$, and *LIML coincides with ILS and 2SLS for an exactly identified equation*.
- For an over-identified equation $\hat{\kappa} > 1$ in general, and LIML differs from 2SLS.

With one endogenous regressor ($m = 1$), the second row of $(W_1 - \kappa W)\mathbf{a} = 0$, with $\mathbf{a} = (1, -\beta)'$, gives

$$\hat{\beta}(\kappa) = \frac{(W_1 - \kappa W)_{12}}{(W_1 - \kappa W)_{22}}.$$

It can be shown that

- $\kappa = 0$ gives OLS (after allowing for X_1),
- $\kappa = 1$ gives 2SLS, and
- $\kappa = \hat{\kappa}$ gives LIML.

Estimators of this form are called **k -class estimators**.

Example 8.4 (A LIML calculation)

An equation with one endogenous regressor and two excluded exogenous variables (so it is over-identified) gives

$$W_1 = \begin{pmatrix} 10 & 4 \\ 4 & 5 \end{pmatrix}, \quad W = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}.$$

Step 1 (determinantal equation).

$$|W_1 - \kappa W| = (10 - 4\kappa)(5 - 2\kappa) - (4 - \kappa)^2.$$

Expanding, $(10 - 4\kappa)(5 - 2\kappa) = 50 - 40\kappa + 8\kappa^2$ and $(4 - \kappa)^2 = 16 - 8\kappa + \kappa^2$, so

$$|W_1 - \kappa W| = 7\kappa^2 - 32\kappa + 34 = 0.$$

Example 8.4 (cont.)

Step 2 (roots).

$$\kappa = \frac{32 \pm \sqrt{32^2 - 4(7)(34)}}{2(7)} = \frac{32 \pm \sqrt{72}}{14} = \frac{16 \pm 3\sqrt{2}}{7},$$

that is, 1.6796 and 2.8918. The smallest is $\hat{\kappa} = 1.6796$.

Step 3 (coefficient). The first row of $(W_1 - \hat{\kappa}W)\mathbf{a} = 0$ with $\mathbf{a} = (1, -\beta)'$ is $(10 - 4\hat{\kappa}) - (4 - \hat{\kappa})\beta = 0$, so

$$\hat{\beta}_{LIML} = \frac{10 - 4\hat{\kappa}}{4 - \hat{\kappa}} = \frac{10 - 6.7185}{2.3204} = \frac{3.2815}{2.3204} = 1.4142.$$

Example 8.4 (cont.)

The second row gives the same value:

$$(4 - \hat{\kappa})/(5 - 2\hat{\kappa}) = 2.3204/1.6408 = 1.4142.$$

Step 4 (check the ratio). With $\alpha = (1, -b)'$,

$$\kappa(b) = (10 - 8b + 5b^2)/(4 - 2b + 2b^2).$$

At $b = 1.4142$:

$$(10 - 11.3137 + 10)/(4 - 2.8284 + 4) = 8.6863/5.1716 = 1.6796 = \hat{\kappa}.$$

At $b = 1$: $7/4 = 1.75 > \hat{\kappa}$.

Comparison (k -class). $\kappa = 0$: $\hat{\beta} = 4/5 = 0.8$ (OLS); $\kappa = 1$: $\hat{\beta} = (4 - 1)/(5 - 2) = 1$ (2SLS); $\kappa = \hat{\kappa}$: $\hat{\beta} = 1.4142$ (LIML).

Practice

- Exercise 8.5 (Exercises and Solutions booklet).

Next lecture

Lecture 60: Properties of LIML; comparison of the methods; summary.