

# Lecture 2: Non-randomised decision rules; the risk function; risk of an estimator and of a test

Statistical Inference–II ■ Unit I: Decision Theory

Kapil Kumar

Department of Statistics  
Chaudhary Charan Singh University, Meerut

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## Previous lecture

Lecture 1: Introduction; elements of a decision problem; standard loss functions.

## Today

- Decision Rules and the Risk Function
  - Non-Randomised Decision Rules
  - The Risk Function

Any decision procedure dictates which action should be taken when  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  is the set of observations, for every possible  $\mathbf{x}$ . In other words, a decision procedure sets up a correspondence between each point of the sample space  $\mathcal{X}$  and some element of the action space  $\mathcal{A}$ .

### Definition 2.1 (Non-randomised decision rule)

A *non-randomised decision rule* is a function

$$\delta : \mathcal{X} \longrightarrow \mathcal{A},$$

where  $\delta(\mathbf{x})$  is the action taken when  $\mathbf{x}$  is observed. (Measurability of  $\delta$  is assumed throughout.)

# The class of decision rules

- The class of all decision rules under consideration is denoted by  $\mathcal{D}$  (some texts use  $\Delta$ ).
- The essential objective of decision theory is to specify a rule in  $\mathcal{D}$  that may be regarded as most desirable.

A criterion for a good decision rule should be based on the loss  $L(\theta, \delta(\mathbf{X}))$  associated with the rule  $\delta$ . However:

- since  $\mathbf{X}$  is random,  $L(\theta, \delta(\mathbf{X}))$  is itself a random variable;
- we cannot ensure that a given rule will always lead to the smallest loss, even if  $\theta$  were known.

Instead we compare rules by their *average* loss.

## Definition 2.2 (Risk function)

The *risk function* of a decision rule  $\delta$  is the expected loss, regarded as a function of  $\theta$ :

$$R(\theta, \delta) = \mathbb{E}_{\theta}[L(\theta, \delta(\mathbf{X}))] = \int_{\mathcal{X}} L(\theta, \delta(\mathbf{x})) f(\mathbf{x} \mid \theta) d\mathbf{x}, \quad \theta \in \Theta,$$

with the integral replaced by a sum in the discrete case.

## Remark 2.3

It is important to be clear about what the risk depends on. The randomness in  $\mathbf{X}$  has been *integrated out*;  $R(\theta, \delta)$  is a non-random function of the two arguments  $\theta$  and  $\delta$ . Comparing two rules therefore means comparing two *functions* on  $\Theta$ , and this is precisely the source of the difficulty: functions are not totally ordered.

### Example 2.4 (Risk of an estimator under SELF)

Under squared error loss  $L(\theta, a) = (a - \theta)^2$ , the risk of an estimator  $\delta(\mathbf{X})$  is

$$R(\theta, \delta) = \mathbb{E}_{\theta}[(\delta(\mathbf{X}) - \theta)^2] = \text{Var}_{\theta}(\delta(\mathbf{X})) + \{b(\theta)\}^2,$$

where  $b(\theta) = \mathbb{E}_{\theta}[\delta(\mathbf{X})] - \theta$  is the bias.

Thus the risk under SELF is exactly the *mean squared error*, and the classical bias–variance decomposition is a statement about risk.

### Example 2.5 (Risk of a test under the testing loss $(c_{01}, c_{10})$ )

Consider testing  $H_0 : \theta \in \Theta_0$  against  $H_1 : \theta \in \Theta_1$  with the testing loss function introduced earlier, and let  $\delta$  be a non-randomised test with critical region  $C = \{\mathbf{x} : \delta(\mathbf{x}) = a_1\}$ . Then

$$R(\theta, \delta) = \begin{cases} c_{01} \mathbb{P}_\theta(\mathbf{X} \in C) = c_{01} \alpha(\theta), & \theta \in \Theta_0, \\ c_{10} \mathbb{P}_\theta(\mathbf{X} \notin C) = c_{10} \beta(\theta), & \theta \in \Theta_1, \end{cases}$$

where  $\alpha(\theta)$  is the probability of a Type I error and  $\beta(\theta)$  that of a Type II error.

The risk function of a test is thus a weighted combination of the two error probabilities. This is the precise sense in which *testing is a decision problem*; the point is developed further in the section “Testing Viewed as a Decision Problem”.

## Practice (Lecture Notes, Chapter 1)

- Short-answer questions (Section 1.12): Q1, Q3, Q4.

## Next lecture

Lecture 3: Randomised decision rules; convexity of the risk set.