Lecture 6: Minimax and Bayes rules among non-randomised rules: the Bernoulli example

Statistical Inference–II ▪ Unit I: Decision Theory

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This lecture

Previous lecture

Lecture 5: The minimax principle; minimax action in a no-data problem.

Today

- The Minimax Principle
  - Worked Example: Minimax in a Finite Problem

Example 6.1 (Minimax and Bayes with a Bernoulli observation)

Let $X \sim \text{Bernoulli}(p)$ with $p \in \Theta = \{p_1, p_2\} = \left\{\frac{1}{4}, \frac{1}{2}\right\}$ and $\mathcal{A} = \{a_1, a_2\}$. The loss function is

$L(p,a)$	$p_1 = \frac{1}{4}$	$p_2 = \frac{1}{2}$
a_1	1	3
a_2	4	2

Since $\mathcal{X} = \{0, 1\}$ and $\mathcal{A}$ has two elements, there are exactly $2^2 = 4$ non-randomised decision rules:

$$\delta_1(0) = a_1, \quad \delta_1(1) = a_1, \qquad \delta_2(0) = a_1, \quad \delta_2(1) = a_2,$$
$$\delta_3(0) = a_2, \quad \delta_3(1) = a_1, \qquad \delta_4(0) = a_2, \quad \delta_4(1) = a_2.$$

Obtain (i) the minimax rule among non-randomised rules, and (ii) the Bayes rule with respect to the prior $g(p_1) = \frac{3}{4},\ g(p_2) = \frac{1}{4}$.

Bernoulli example: risks at $p_1 = \frac{1}{4}$

Solution

The risk is

$$R(p, \delta) = \mathbb{E}_p[L(p, \delta(X))] = L(p, \delta(0))\,(1-p) + L(p, \delta(1))\,p.$$

Risks at $p_1 = \frac{1}{4}$, so $1 - p_1 = \frac{3}{4}$:

$$\begin{aligned}
R(p_1, \delta_1) &= L(p_1, a_1) \cdot \tfrac{3}{4} + L(p_1, a_1) \cdot \tfrac{1}{4} = 1 \cdot \tfrac{3}{4} + 1 \cdot \tfrac{1}{4} = 1, \\
R(p_1, \delta_2) &= L(p_1, a_1) \cdot \tfrac{3}{4} + L(p_1, a_2) \cdot \tfrac{1}{4} = 1 \cdot \tfrac{3}{4} + 4 \cdot \tfrac{1}{4} = \tfrac{3}{4} + 1 = \tfrac{7}{4}, \\
R(p_1, \delta_3) &= L(p_1, a_2) \cdot \tfrac{3}{4} + L(p_1, a_1) \cdot \tfrac{1}{4} = 4 \cdot \tfrac{3}{4} + 1 \cdot \tfrac{1}{4} = 3 + \tfrac{1}{4} = \tfrac{13}{4}, \\
R(p_1, \delta_4) &= L(p_1, a_2) \cdot \tfrac{3}{4} + L(p_1, a_2) \cdot \tfrac{1}{4} = 4 \cdot \tfrac{3}{4} + 4 \cdot \tfrac{1}{4} = 4.
\end{aligned}$$

Bernoulli example: risks at $p_2 = \frac{1}{2}$; minimax rule

Solution (cont.)

Risks at $p_2 = \frac{1}{2}$, so $1 - p_2 = \frac{1}{2}$:

$$R(p_2, \delta_1) = 3 \cdot \tfrac{1}{2} + 3 \cdot \tfrac{1}{2} = 3, \qquad R(p_2, \delta_2) = 3 \cdot \tfrac{1}{2} + 2 \cdot \tfrac{1}{2} = \tfrac{5}{2},$$
$$R(p_2, \delta_3) = 2 \cdot \tfrac{1}{2} + 3 \cdot \tfrac{1}{2} = \tfrac{5}{2}, \qquad R(p_2, \delta_4) = 2 \cdot \tfrac{1}{2} + 2 \cdot \tfrac{1}{2} = 2.$$

(i) Minimax rule.

j	$R(p_1, \delta_j)$	$R(p_2, \delta_j)$	$\max_p R(p, \delta_j)$
1	1	3	3
2	$\frac{7}{4}$	$\frac{5}{2}$	$\frac{5}{2}$
3	$\frac{13}{4}$	$\frac{5}{2}$	$\frac{13}{4}$
4	4	2	4

Solution (cont.)

The smallest maximum risk is $\frac{5}{2}$, attained by δ_2. Hence, within the class of non-randomised rules,

$$\delta_2(x) = \begin{cases} a_1, & x = 0, \\ a_2, & x = 1, \end{cases}$$

is the minimax rule, with minimax risk $\frac{5}{2}$.

Admissibility check. Comparing risk pairs

$$(1, 3), \quad \left(\tfrac{7}{4}, \tfrac{5}{2}\right), \quad \left(\tfrac{13}{4}, \tfrac{5}{2}\right), \quad (4, 2),$$

we see that δ_2 dominates δ_3 (since $\frac{7}{4} < \frac{13}{4}$ and $\frac{5}{2} \leq \frac{5}{2}$), so δ_3 is inadmissible. The rules $\delta_1, \delta_2, \delta_4$ are mutually incomparable and hence all admissible within this class.

Solution (cont.)

(ii) Bayes rule. With $g(p_1) = \frac{3}{4}$, $g(p_2) = \frac{1}{4}$ the Bayes risk of δ_j is

$$r(g, \delta_j) = \tfrac{3}{4} R(p_1, \delta_j) + \tfrac{1}{4} R(p_2, \delta_j) :$$

$$\begin{aligned}
r(g, \delta_1) &= \tfrac{3}{4}(1) + \tfrac{1}{4}(3) = \tfrac{3}{4} + \tfrac{3}{4} = \tfrac{3}{2} = \tfrac{24}{16}, \\
r(g, \delta_2) &= \tfrac{3}{4} \cdot \tfrac{7}{4} + \tfrac{1}{4} \cdot \tfrac{5}{2} = \tfrac{21}{16} + \tfrac{10}{16} = \tfrac{31}{16}, \\
r(g, \delta_3) &= \tfrac{3}{4} \cdot \tfrac{13}{4} + \tfrac{1}{4} \cdot \tfrac{5}{2} = \tfrac{39}{16} + \tfrac{10}{16} = \tfrac{49}{16}, \\
r(g, \delta_4) &= \tfrac{3}{4} \cdot 4 + \tfrac{1}{4} \cdot 2 = 3 + \tfrac{1}{2} = \tfrac{7}{2} = \tfrac{56}{16}.
\end{aligned}$$

Bernoulli example: the Bayes rule (cont.)

Solution (cont.)

j	$r(g, \delta_j)$
1	$24/16 = 3/2$
2	$31/16$
3	$49/16$
4	$56/16 = 7/2$

The minimum Bayes risk is $\frac{3}{2}$, so δ_1 is the Bayes rule with respect to g.

Bayes rule versus minimax rule

Remark 6.2

Notice that the Bayes rule (δ_1) and the non-randomised minimax rule (δ_2) differ. This is typical: the Bayes rule exploits the prior belief that $p = \frac{1}{4}$ is three times as likely as $p = \frac{1}{2}$, whereas the minimax rule ignores this and protects against the worse state.

- In general a Bayes rule need not be minimax. Conditions under which a Bayes rule is minimax are given in "Least Favourable Priors and Minimax Rules".
- The example is continued in "Randomised Minimax Rules in Finite Problems", where the minimax rule over *randomised* rules is found.

After this lecture

Practice (Lecture Notes, Chapter 1)

Re-work the examples of this lecture without looking at the solutions.

Next lecture

Lecture 7: Prior distribution, Bayes risk and Bayes rule.