

Lecture 10: Small-neighbourhood (MAP) and LINEX losses

Statistical Inference–II ■ Unit I: Decision Theory

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Previous lecture

Lecture 9: Bayes estimators under squared error, weighted squared error and absolute error loss.

Today

- The Bayes Principle
 - Bayes Rules under Standard Loss Functions

(iii) Small-neighbourhood 0–1 loss

Corollary 10.1 (Bayes estimator under small-neighbourhood loss)

For

$$L_{\epsilon}(\theta, a) = \begin{cases} 0, & |a - \theta| \leq \epsilon, \\ 1, & |a - \theta| > \epsilon, \end{cases}$$

the Bayes estimator is the action a that maximises the posterior probability

$$Q(a) = \mathbb{P}(a - \epsilon \leq \theta \leq a + \epsilon \mid \mathbf{x}).$$

When ϵ is small and the posterior density is continuous,

$Q(a) \approx 2\epsilon \pi(a \mid \mathbf{x})$, so the Bayes estimate is close to the posterior mode $\tilde{\theta}(\mathbf{x})$, the value of θ that maximises $\pi(\theta \mid \mathbf{x})$. The posterior mode is called the maximum a posteriori (MAP) estimator.

Proof.

The posterior expected loss of the action a is

$$\rho(\pi, a \mid \mathbf{x}) = 1 \cdot \mathbb{P}(|a - \theta| > \epsilon \mid \mathbf{x}) + 0 \cdot \mathbb{P}(|a - \theta| \leq \epsilon \mid \mathbf{x}) = 1 - Q(a),$$

which is minimised by maximising $Q(a)$. For small ϵ ,

$$Q(a) = \int_{a-\epsilon}^{a+\epsilon} \pi(\theta \mid \mathbf{x}) d\theta \approx 2\epsilon \pi(a \mid \mathbf{x}),$$

which is largest when a is at (or near) the mode. □

Remark 10.2

- The approximation in the last step is heuristic. For a unimodal posterior it can be made exact: every Bayes action lies within ϵ of the mode and so tends to the mode as $\epsilon \downarrow 0$ (Lecture Notes, Appendix A).
- Note also that the literal loss $L(\theta, a) = \mathbf{1}(a \neq \theta)$ is of no use for a continuously distributed parameter: it gives posterior expected loss one for every action. The MAP estimator must therefore be justified through L_ϵ .

Corollary 10.3 (Bayes estimator under LINEX loss)

Let $L(\theta, a) = e^{c(a-\theta)} - c(a - \theta) - 1$, $c \neq 0$, and suppose that $\mathbb{E}(e^{-c\theta} \mid \mathbf{x})$ and $\mathbb{E}(|\theta| \mid \mathbf{x})$ are finite. Then the Bayes estimator is

$$\delta_L(\mathbf{x}) = -\frac{1}{c} \log \mathbb{E}(e^{-c\theta} \mid \mathbf{x}).$$

Proof.

For fixed $\mathbf{x}$,

$$h(a) = \mathbb{E}\{e^{c(a-\theta)} - c(a-\theta) - 1 \mid \mathbf{x}\} = e^{ca} \mathbb{E}(e^{-c\theta} \mid \mathbf{x}) - ca + c \mathbb{E}(\theta \mid \mathbf{x}) - 1.$$

Then

$$\begin{aligned} h'(a) = c e^{ca} \mathbb{E}(e^{-c\theta} \mid \mathbf{x}) - c = 0 &\iff e^{ca} = \frac{1}{\mathbb{E}(e^{-c\theta} \mid \mathbf{x})} \\ &\iff a = -\frac{1}{c} \log \mathbb{E}(e^{-c\theta} \mid \mathbf{x}), \end{aligned}$$

and $h''(a) = c^2 e^{ca} \mathbb{E}(e^{-c\theta} \mid \mathbf{x}) > 0$ confirms a minimum. □

Remark 10.4

Since $e^{-c\theta}$ is a convex function of θ , Jensen's inequality gives $\mathbb{E}(e^{-c\theta} \mid \mathbf{x}) \geq e^{-c\mathbb{E}(\theta \mid \mathbf{x})}$. Hence

$$\delta_L(\mathbf{x}) \leq \mathbb{E}(\theta \mid \mathbf{x}) \text{ when } c > 0, \quad \delta_L(\mathbf{x}) \geq \mathbb{E}(\theta \mid \mathbf{x}) \text{ when } c < 0.$$

- With $c > 0$ over-estimation is the more serious error, and the Bayes estimate is pulled below the posterior mean; with $c < 0$ the reverse.
- As $c \rightarrow 0$, $\delta_L(\mathbf{x}) \rightarrow \mathbb{E}(\theta \mid \mathbf{x})$.
- If the posterior is $N(m, v)$, then $\mathbb{E}(e^{-c\theta} \mid \mathbf{x}) = e^{-cm + c^2v/2}$ and $\delta_L = m - cv/2$.

Remark 10.5

Four familiar summaries of the posterior distribution thus arise as Bayes estimators:

- the mean under squared error,
- the median under absolute error,
- the mode (approximately) under small-neighbourhood 0–1 loss, and
- $-c^{-1} \log \mathbb{E}(e^{-c\theta} \mid \mathbf{x})$ under LINEX loss.

Examples 12.5 and 12.6 show that they can differ considerably.

Practice (Lecture Notes, Chapter 1)

- Short-answer questions (Section 1.12): Q10.
- Exercises (Section 1.13): 1.4, 1.16.

Next lecture

Lecture 11: Worked examples: linear rules; the beta–binomial model.