

Lecture 13: Bayes rules and admissibility

Statistical Inference–II ■ Unit I: Decision Theory

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Previous lecture

Lecture 12: Normal–normal and Poisson–gamma models; different losses, different estimates.

Today

- Relations among Bayes, Minimax and Admissible Rules
 - Bayes Rules and Admissibility

We now establish the principal theorems connecting the three concepts. These form the theoretical core of Unit I.

Theorem 13.1 (A unique Bayes rule is admissible)

Let δ_g be a Bayes rule with respect to a prior g , and suppose that δ_g is unique up to risk equivalence, in the sense that any other Bayes rule δ' with respect to g satisfies $R(\theta, \delta') = R(\theta, \delta_g)$ for all θ . Then δ_g is admissible.

Proof: A unique Bayes rule is admissible

Proof.

Since δ_g is Bayes with respect to g ,

$$r(g, \delta_g) \leq r(g, \delta) \quad \text{for every } \delta \in \mathcal{D}. \quad (13.1)$$

Suppose, if possible, that δ_g is *not* admissible. Then there exists a rule $\delta' \in \mathcal{D}$ such that

$$R(\theta, \delta') \leq R(\theta, \delta_g) \quad \text{for all } \theta \in \Theta, \quad (13.2)$$

and

$$R(\theta_0, \delta') < R(\theta_0, \delta_g) \quad \text{for at least one } \theta_0 \in \Theta. \quad (13.3)$$

Proof (cont.)

Multiplying (13.2) by $g(\theta) \geq 0$ and integrating over Θ ,

$$\int_{\Theta} R(\theta, \delta') g(\theta) d\theta \leq \int_{\Theta} R(\theta, \delta_g) g(\theta) d\theta,$$

that is,

$$r(g, \delta') \leq r(g, \delta_g). \quad (13.4)$$

Combining (13.1) and (13.4) we obtain $r(g, \delta') = r(g, \delta_g)$, so that δ' is *also* a Bayes rule with respect to g .

Proof: A unique Bayes rule is admissible (cont.)

Proof (cont.)

By the uniqueness hypothesis (up to risk equivalence), we must then have

$$R(\theta, \delta') = R(\theta, \delta_g) \quad \text{for all } \theta \in \Theta,$$

which contradicts the strict inequality (13.3). Hence no such δ' exists and δ_g is admissible. $\square$

Remark 13.2 (Why uniqueness is needed)

Without the uniqueness assumption the conclusion may fail. If two Bayes rules have the same Bayes risk but different risk functions, one may dominate the other, and the dominated one — though Bayes — is inadmissible.

Conditions that guarantee uniqueness, and a counterexample showing that uniqueness of the posterior mean at each x is not quite enough, are given in Lecture Notes, Appendix A.

Theorem 13.3 (Bayes with respect to a positive prior on a finite Θ)

Let $\Theta = \{\theta_1, \dots, \theta_k\}$ be finite and let g assign positive probability to every θ_i . Then every Bayes rule with respect to g is admissible.

Proof.

Let δ_g be Bayes and suppose δ' dominates it, so that (13.2) and (13.3) hold with $\theta_0 = \theta_{i_0}$, say. Then

$$r(g, \delta_g) - r(g, \delta') = \sum_{i=1}^k \{R(\theta_i, \delta_g) - R(\theta_i, \delta')\} g(\theta_i).$$

Every summand is non-negative by (13.2), and the summand for $i = i_0$ is strictly positive because $g(\theta_{i_0}) > 0$ and the risk difference is strictly positive. Hence $r(g, \delta_g) > r(g, \delta')$, contradicting the Bayes optimality of δ_g . □

Theorem 13.4 (Admissibility of Bayes rules under continuity)

Let Θ be an interval, let the prior density $g(\theta) > 0$ for all $\theta \in \Theta$, and suppose $R(\theta, \delta)$ is continuous in θ for every $\delta \in \mathcal{D}$. If δ_g is a Bayes rule with $r(g, \delta_g) < \infty$, then δ_g is admissible.

Proof.

Suppose δ' dominates δ_g , with strict inequality at θ_0 . Set $\Delta(\theta) = R(\theta, \delta_g) - R(\theta, \delta') \geq 0$. By continuity, $\Delta(\theta_0) = \epsilon > 0$ implies there is an $h > 0$ such that $\Delta(\theta) > \epsilon/2$ on $I_h = (\theta_0 - h, \theta_0 + h) \cap \Theta$. (If θ_0 is an end-point of Θ , I_h is a one-sided interval; it still has positive length.)

Proof (cont.)

Then

$$\begin{aligned} r(g, \delta_g) - r(g, \delta') &= \int_{\Theta} \Delta(\theta) g(\theta) d\theta \geq \int_{I_h} \frac{\epsilon}{2} g(\theta) d\theta \\ &= \frac{\epsilon}{2} \mathbb{P}_g(\theta \in I_h) > 0, \end{aligned}$$

the last inequality because $g > 0$ on I_h , which has positive length. This contradicts Bayes optimality. □

Practice (Lecture Notes, Chapter 1)

Re-work the examples of this lecture without looking at the solutions.

Next lecture

Lecture 14: Least favourable priors; Bayes rules with constant risk are minimax.