

Lecture 14: Least favourable priors; Bayes rules with constant risk are minimax

Statistical Inference–II ■ Unit I: Decision Theory

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M.Sc. Statistics, Semester III
Academic Session 2026–2027

Previous lecture

Lecture 13: Bayes rules and admissibility.

Today

- Relations among Bayes, Minimax and Admissible Rules
 - Least Favourable Priors and Minimax Rules

Definition 14.1 (Least favourable prior)

A prior distribution g is said to be *least favourable* if

$$r(g) \geq r(g') \quad \text{for every other prior } g',$$

where $r(g) = \inf_{\delta \in \mathcal{D}} r(g, \delta)$ is the Bayes risk of the problem under g . In words, g is the prior which makes the problem as hard as possible in the Bayes sense.

Theorem 14.2 (Fundamental theorem relating Bayes and minimax)

Let g be a prior distribution and let δ_g be a Bayes rule with respect to g . If

$$r(g, \delta_g) \geq \sup_{\theta \in \Theta} R(\theta, \delta_g), \quad (14.1)$$

then

- (a) δ_g is a minimax rule;*
- (b) g is a least favourable prior distribution.*

Proof.

First observe that the Bayes risk, being an average of the risk function, never exceeds the supremum of the risk function:

$$\begin{aligned} r(g, \delta_g) &= \int_{\Theta} R(\theta, \delta_g) g(\theta) d\theta \leq \int_{\Theta} \left[\sup_{\theta'} R(\theta', \delta_g) \right] g(\theta) d\theta \\ &= \sup_{\theta} R(\theta, \delta_g), \end{aligned}$$

since g integrates to one. Combining this with the hypothesis (14.1) we obtain the key equality

$$r(g, \delta_g) = \sup_{\theta \in \Theta} R(\theta, \delta_g). \quad (14.2)$$

Proof (cont.)

(a) δ_g is minimax. Let δ be any rule in $\mathcal{D}$. Then

$$\begin{aligned}\sup_{\theta} R(\theta, \delta) &\geq \int_{\Theta} R(\theta, \delta) g(\theta) d\theta = r(g, \delta) \\ &\geq r(g, \delta_g) = \sup_{\theta} R(\theta, \delta_g),\end{aligned}$$

where the first inequality is again “average $\leq$ supremum”, the second is the Bayes optimality of δ_g , and the final equality is (14.2). Since δ was arbitrary, δ_g minimises the maximum risk and is therefore minimax.

Proof (cont.)

(b) g is least favourable. Let g' be any other prior (a Bayes rule with respect to g' need not exist). Then

$$\begin{aligned} r(g') = \inf_{\delta \in \mathcal{D}} r(g', \delta) &\leq r(g', \delta_g) = \int_{\Theta} R(\theta, \delta_g) g'(\theta) d\theta \\ &\leq \sup_{\theta} R(\theta, \delta_g) = r(g, \delta_g) = r(g). \end{aligned}$$

The first inequality holds because $r(g')$ is an infimum over $\mathcal{D}$ and $\delta_g \in \mathcal{D}$; the second is again “average $\leq$ supremum”; the final equality is (14.2). Thus $r(g') \leq r(g)$ for every prior g' , and g is least favourable. $\square$

Corollary 14.3 (Bayes rule with constant risk is minimax)

If δ_g is a Bayes rule with respect to some prior g and its risk function $R(\theta, \delta_g)$ is constant over Θ , then δ_g is minimax and g is least favourable.

Proof.

If $R(\theta, \delta_g) \equiv C$ then $r(g, \delta_g) = \int C g(\theta) d\theta = C = \sup_{\theta} R(\theta, \delta_g)$, so the hypothesis (14.1) of Theorem 14.2 holds with equality. $\square$

Corollary 14.3 is the standard route to minimax estimators: *find a prior for which the Bayes rule happens to have constant risk*. Such a rule is called an *equaliser rule*.

Corollary 14.4 (Unique minimax rules are admissible)

If a minimax rule is unique, then it is admissible.

Proof.

Let δ_M be the unique minimax rule and suppose it is inadmissible, so some δ' satisfies $R(\theta, \delta') \leq R(\theta, \delta_M)$ for all θ with strict inequality somewhere.

Taking suprema,

$$\sup_{\theta} R(\theta, \delta') \leq \sup_{\theta} R(\theta, \delta_M),$$

so δ' is also minimax. By uniqueness $\delta' = \delta_M$ (as decision rules, or at least in risk), contradicting the strict inequality. $\square$

Practice (Lecture Notes, Chapter 1)

- Short-answer questions (Section 1.12): Q11, Q12.
- Exercises (Section 1.13): 1.10, 1.17.

Next lecture

Lecture 15: Randomised minimax rules in finite problems.