Lecture 15: Randomised minimax rules in finite problems

Statistical Inference–II ▪ Unit I: Decision Theory

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This lecture

Previous lecture

Lecture 14: Least favourable priors; Bayes rules with constant risk are minimax.

Today

- Relations among Bayes, Minimax and Admissible Rules
  - Randomised Minimax Rules in Finite Problems

Randomised minimax rules in finite problems

When Θ is finite, Corollary 14.3 gives a practical recipe for finding a minimax rule:

look for a (possibly randomised) rule whose risk is the same at every state, and check that it is Bayes with respect to some prior.

We apply the recipe to the two finite examples of "The Minimax Principle": the no-data problem and the Bernoulli example.

No-data example continued: a randomised minimax action

Example 15.1 (No-data minimax continued: a randomised minimax action)

In the no-data problem with loss table

$L(\theta, a)$	θ_1	θ_2
a_1	20	35
a_2	25	15

the best non-randomised action a_2 has maximum loss 25. Find the minimax rule when randomisation is allowed, and the least favourable prior.

No-data example continued: the equaliser rule

Solution

Take a_1 with probability λ and a_2 with probability $1-\lambda$. By the proposition "Randomisation convexifies the risk set" the expected losses are

$$\theta_1:\ 20\lambda + 25(1-\lambda) = 25 - 5\lambda, \qquad \theta_2:\ 35\lambda + 15(1-\lambda) = 15 + 20\lambda.$$

They are equal when $25 - 5\lambda = 15 + 20\lambda$, i.e. $\lambda = \frac{2}{5}$. The rule

$$\text{take } a_1 \text{ with probability } \tfrac{2}{5}, \qquad a_2 \text{ with probability } \tfrac{3}{5},$$

has expected loss $25 - 5 \cdot \frac{2}{5} = 23$ under θ_1 and $15 + 20 \cdot \frac{2}{5} = 23$ under θ_2: it is an equaliser rule.

No-data example continued: the least favourable prior

Solution (cont.)

To show that it is minimax we find a prior under which it is Bayes. With $g(\theta_1) = w$, the Bayes risks of the two actions are

$$r(g, a_1) = 20w + 35(1 - w) = 35 - 15w,$$
$$r(g, a_2) = 25w + 15(1 - w) = 15 + 10w.$$

These are equal when $35 - 15w = 15 + 10w$, i.e. $w = \frac{4}{5}$, and both then equal 23.

Under the prior $g(\theta_1) = \frac{4}{5}$, $g(\theta_2) = \frac{1}{5}$, every mixture of a_1 and a_2 has Bayes risk 23, so the equaliser rule is Bayes with constant risk. By Corollary 14.3 it is minimax and the prior is least favourable.

The minimax value over randomised rules is $23 < 25$.

Bernoulli example continued

Example 15.2 (Bernoulli example continued: Bayes, least favourable and randomised minimax rules)

In the Bernoulli example (Example 6.1),

(i) find, for every prior, which non-randomised rule is Bayes; and

(ii) find the minimax rule when randomised rules are allowed, and the least favourable prior.

Recall the risks of the four non-randomised rules:

j	1	2	3	4
$R(p_1, \delta_j)$	1	$\frac{7}{4}$	$\frac{13}{4}$	4
$R(p_2, \delta_j)$	3	$\frac{5}{2}$	$\frac{5}{2}$	2

Bernoulli example continued: (i) Bayes rules for every prior

Solution

(i) Bayes rules for every prior. Write $w = g(p_1)$. From the risk table,

$$r(g,\delta_1) = 3 - 2w, \qquad r(g,\delta_2) = \tfrac{5}{2} - \tfrac{3}{4}w,$$
$$r(g,\delta_3) = \tfrac{5}{2} + \tfrac{3}{4}w, \qquad r(g,\delta_4) = 2 + 2w.$$

The rule δ_3 is never Bayes, being dominated by δ_2. Comparing the others,

- δ_2 is at least as good as δ_1 when $\frac{5}{2} - \frac{3}{4}w \le 3 - 2w$, i.e. $w \le \frac{2}{5}$;
- δ_2 is at least as good as δ_4 when $\frac{5}{2} - \frac{3}{4}w \le 2 + 2w$, i.e. $w \ge \frac{2}{11}$;
- δ_4 beats δ_1 exactly when $w \le \frac{1}{4}$.

Bernoulli example continued: (i) Bayes rules for every prior

Solution (cont.)

Hence

$$\text{the Bayes rule is} \quad \begin{cases} \delta_4, & 0 \le w \le \frac{2}{11}, \\ \delta_2, & \frac{2}{11} \le w \le \frac{2}{5}, \\ \delta_1, & \frac{2}{5} \le w \le 1. \end{cases}$$

For $w = \frac{3}{4}$ this gives δ_1, as found in Example 6.1.

Note that δ_2, the non-randomised minimax rule, is Bayes for a whole interval of priors; being Bayes for some prior does not make that prior least favourable.

Bernoulli example continued: (ii) randomised minimax rule

Solution (cont.)

(ii) Randomised minimax rule. The risk points of the four rules are

$$(1,3), \quad (\tfrac{7}{4},\tfrac{5}{2}), \quad (\tfrac{13}{4},\tfrac{5}{2}), \quad (4,2).$$

The maximum risk $\frac{5}{2}$ of δ_2 occurs at p_2, where δ_4 does better, so we mix δ_2 (probability λ) with δ_4 (probability $1-\lambda$). By the proposition "Randomisation convexifies the risk set",

$$R_1(\lambda) = \tfrac{7}{4}\lambda + 4(1-\lambda) = 4 - \tfrac{9}{4}\lambda, \qquad R_2(\lambda) = \tfrac{5}{2}\lambda + 2(1-\lambda) = 2 + \tfrac{1}{2}\lambda.$$

Equating, $4 - \frac{9}{4}\lambda = 2 + \frac{1}{2}\lambda$, so $\lambda = \frac{8}{11}$ and

$$\boxed{\delta_M^* = \tfrac{8}{11}\,\delta_2 + \tfrac{3}{11}\,\delta_4}$$

has constant risk $R_1 = R_2 = 2 + \frac{1}{2}\cdot\frac{8}{11} = \frac{26}{11} < \frac{5}{2}$.

Bernoulli example continued: (ii) least favourable prior

Solution (cont.)

At $w = \frac{2}{11}$ both δ_2 and δ_4 are Bayes, with common Bayes risk

$$\tfrac{5}{2} - \tfrac{3}{4} \cdot \tfrac{2}{11} = 2 + 2 \cdot \tfrac{2}{11} = \tfrac{26}{11},$$

so every mixture of them, in particular δ^*_M, is Bayes for this prior.

Being Bayes with constant risk, δ^*_M is minimax (Corollary 14.3), and

$$g(p_1) = \tfrac{2}{11}, \qquad g(p_2) = \tfrac{9}{11}$$

is the least favourable prior. Randomisation thus lowers the minimax risk from $\frac{5}{2}$ to $\frac{26}{11}$.

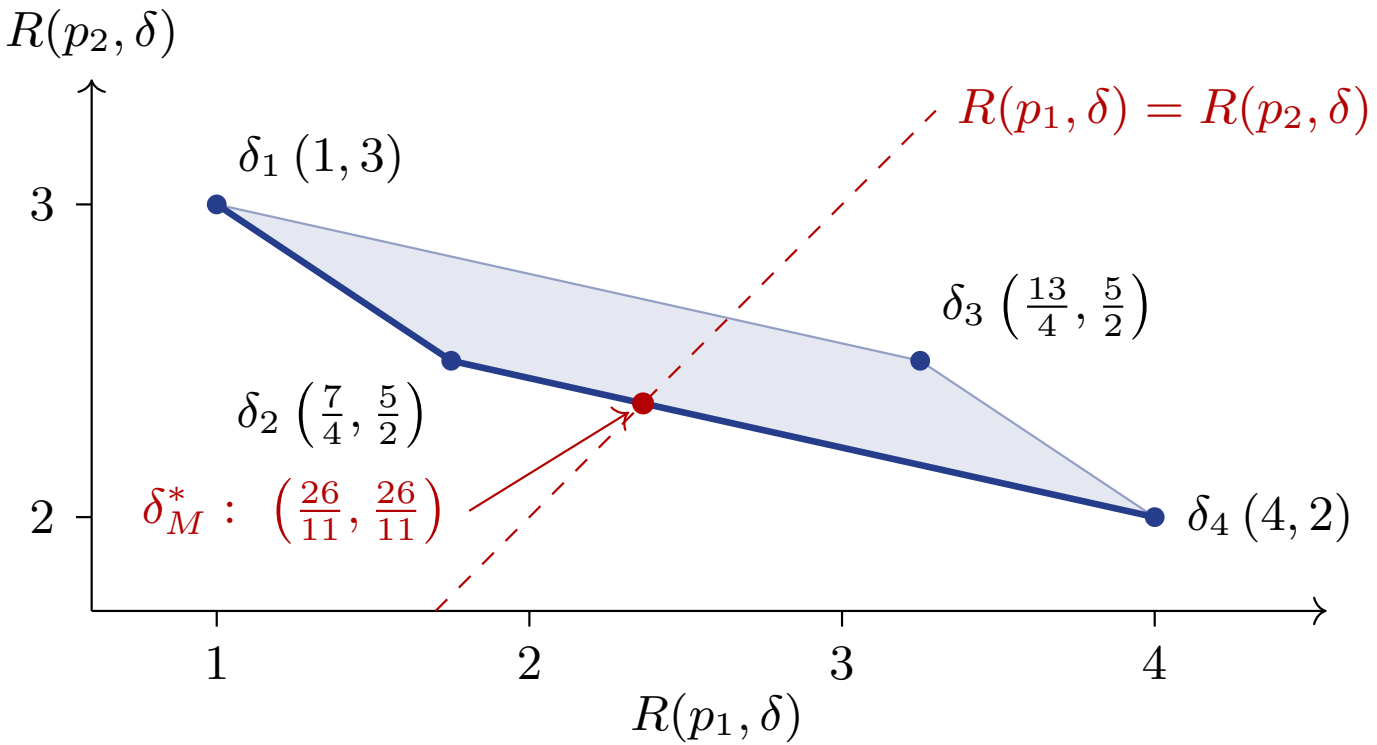
The geometry of the Bernoulli example

Remark 15.3 (The geometry of Example 15.2)

In the plane with coordinates $(R_1, R_2) = (R(p_1, \cdot), R(p_2, \cdot))$ the randomised rules fill the convex hull of the four risk points (the proposition "Randomisation convexifies the risk set").

- The minimax rule is the point of this set on the line $R_1 = R_2$ that is closest to the origin, here $(\frac{26}{11}, \frac{26}{11})$ on the segment joining δ_2 and δ_4.
- That segment has slope $(2 - \frac{5}{2})/(4 - \frac{7}{4}) = -\frac{2}{9}$.
- The Bayes rules for the prior $(w, 1-w)$ minimise $wR_1 + (1-w)R_2$, a family of lines of slope $-w/(1-w)$; the prior is least favourable when these lines are parallel to the segment, i.e. $w/(1-w) = \frac{2}{9}$, which again gives $w = \frac{2}{11}$.

The risk set of the Bernoulli example (sketch)



Shaded: convex hull of the four risk points (risk set of all randomised rules). Thick line: its lower boundary $\delta_1\delta_2\delta_4$. The 45° equaliser line meets the segment $\delta_2\delta_4$ (slope $-2/9$) at $\left(\frac{26}{11}, \frac{26}{11}\right)$, the risk point of $\delta_M^* = \frac{8}{11}\delta_2 + \frac{3}{11}\delta_4$.

After this lecture

Practice (Lecture Notes, Chapter 1)

- Exercises (Section 1.13): 1.20.

Next lecture

Lecture 16: Minimax estimators of a binomial proportion and of a normal mean.