

Lecture 16: Minimax estimators of a binomial proportion and of a normal mean

Statistical Inference–II ■ Unit I: Decision Theory

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Previous lecture

Lecture 15: Randomised minimax rules in finite problems.

Today

- Relations among Bayes, Minimax and Admissible Rules
 - Worked Example: A Minimax Estimator of a Binomial Proportion
 - Minimax Estimation of a Normal Mean

Example 16.1 (Minimax estimator of p)

Let $X \sim \text{Binomial}(n, p)$, $0 \leq p \leq 1$, and let the loss be squared error. Find a minimax estimator of p . (This classical example is due to Hodges and Lehmann, 1950.)

Solution

We use Corollary 14.3: seek a $\text{Beta}(\alpha, \beta)$ prior whose Bayes estimator has constant risk.

Solution (cont.)

Step 1: Bayes estimator under a Beta prior. Let $g(p) \propto p^{\alpha-1}(1-p)^{\beta-1}$. Then, exactly as in the beta-binomial example,

$$\pi(p \mid x) \sim \text{Beta}(x + \alpha, n - x + \beta),$$

so the Bayes estimator under SELF is

$$\delta_{\alpha,\beta}(x) = \frac{x + \alpha}{n + \alpha + \beta}.$$

Solution (cont.)

Step 2: risk function. Write $N = n + \alpha + \beta$. Then

$$\begin{aligned} R(p, \delta_{\alpha, \beta}) &= \text{Var}_p\left(\frac{X + \alpha}{N}\right) + \left\{ \mathbb{E}_p\left(\frac{X + \alpha}{N}\right) - p \right\}^2 \\ &= \frac{np(1-p)}{N^2} + \left\{ \frac{np + \alpha - Np}{N} \right\}^2 \\ &= \frac{np(1-p)}{N^2} + \frac{\{\alpha - (\alpha + \beta)p\}^2}{N^2} \end{aligned}$$

Solution (cont.)

Continuing,

$$\begin{aligned} R(p, \delta_{\alpha, \beta}) &= \frac{1}{N^2} \left[np - np^2 + \alpha^2 - 2\alpha(\alpha + \beta)p + (\alpha + \beta)^2 p^2 \right] \\ &= \frac{1}{N^2} \left[\underbrace{\{(\alpha + \beta)^2 - n\}}_{\text{coeff. of } p^2} p^2 + \underbrace{\{n - 2\alpha(\alpha + \beta)\}}_{\text{coeff. of } p} p + \alpha^2 \right]. \end{aligned}$$

Solution (cont.)

Step 3: impose constant risk. The risk is free of p if and only if both coefficients vanish:

$$(\alpha + \beta)^2 - n = 0 \quad \text{and} \quad n - 2\alpha(\alpha + \beta) = 0.$$

From the first, $\alpha + \beta = \sqrt{n}$. Substituting into the second, $n = 2\alpha\sqrt{n}$, so

$$\alpha = \frac{\sqrt{n}}{2}, \quad \beta = \sqrt{n} - \alpha = \frac{\sqrt{n}}{2}.$$

Minimax Estimator of p : Step 4 (The Minimax Estimator)

Solution (cont.)

Step 4: the minimax estimator. With $\alpha = \beta = \sqrt{n}/2$ and $N = n + \sqrt{n}$,

$$\delta_M(X) = \frac{X + \frac{1}{2}\sqrt{n}}{n + \sqrt{n}} = \frac{X + \frac{1}{2}\sqrt{n}}{\sqrt{n}(\sqrt{n} + 1)}$$

and its constant risk is

$$R(p, \delta_M) = \frac{\alpha^2}{N^2} = \frac{n/4}{n(\sqrt{n} + 1)^2} = \frac{1}{4(1 + \sqrt{n})^2}.$$

Since δ_M is Bayes with respect to $\text{Beta}(\sqrt{n}/2, \sqrt{n}/2)$ and has constant risk, it is minimax by Corollary 14.3, and $\text{Beta}(\sqrt{n}/2, \sqrt{n}/2)$ is a least favourable prior.

Remark 16.2 (Comparison with the sample proportion)

The usual estimator $\bar{p} = X/n$ has risk $p(1-p)/n$, with maximum $1/(4n)$ at $p = \frac{1}{2}$. Since

$$\frac{1}{4(1 + \sqrt{n})^2} < \frac{1}{4n} \quad \text{for all } n \geq 1,$$

the minimax estimator has strictly smaller maximum risk. However, $R(p, \bar{p}) < R(p, \delta_M)$ for p near 0 or 1, so $\bar{p}$ is *not* dominated by δ_M ; the two are incomparable, and both are admissible (δ_M as a unique Bayes rule; for the admissibility of X/n see Lehmann and Casella, 1998, Ch. 5). This is the standard illustration of the trade-off inherent in the minimax criterion.

Remark 16.3 (Is Laplace's estimator minimax?)

Return to the estimator $\delta_g(X) = (X + 1)/(n + 2)$ of the beta-binomial example (Example 11.3), whose risk is $\{1 + (n - 4)p(1 - p)\}/(n + 2)^2$. Since $0 \leq p(1 - p) \leq \frac{1}{4}$,

$$\sup_p R(p, \delta_g) = \frac{\max\{1, n/4\}}{(n + 2)^2} > \frac{1}{4(1 + \sqrt{n})^2} = \sup_p R(p, \delta_M) \quad (n \neq 4),$$

because

- for $n < 4$ the inequality reduces to $n + 2 < 2 + 2\sqrt{n}$, i.e. $\sqrt{n} < 2$, and
- for $n > 4$ to $n + 2 < \sqrt{n}(1 + \sqrt{n})$, i.e. $\sqrt{n} > 2$.

Remark (cont.)

- Hence δ_g is *not* minimax when $n \neq 4$.
- For $n = 4$, on the other hand, $\delta_g = (X + 1)/6 = \delta_M$ has constant risk $1/36$: the uniform prior $\text{Beta}(1, 1) = \text{Beta}(\sqrt{n}/2, \sqrt{n}/2)$ is then least favourable (Exercise 1.21).
- That δ_M is the *only* minimax estimator is shown in Lecture Notes, Appendix A.

Example 16.4 ($\bar{X}$ is minimax for a normal mean)

Let $X_1, \dots, X_n$ be i.i.d. $N(\mu, \sigma^2)$ with σ^2 known and $\mu \in \mathbb{R}$. Under squared error loss, show that $\bar{X}$ is minimax.

Solution

The rule $\delta_0(\mathbf{X}) = \bar{X}$ has constant risk $R(\mu, \delta_0) = \sigma^2/n$, but it is not Bayes with respect to any proper prior (see Exercise 1.9). We therefore use a *limiting* argument.

$\bar{X}$ is Minimax for a Normal Mean (cont.)

Solution (cont.)

Take the prior $\mu \sim N(0, \tau^2)$. By the normal-normal example (adapted for known σ^2), the Bayes rule is $\delta_\tau(\mathbf{X}) = \frac{n\tau^2}{n\tau^2 + \sigma^2} \bar{X}$ with Bayes risk

$$r(\tau) = \frac{\sigma^2 \tau^2}{n\tau^2 + \sigma^2} = \frac{\sigma^2}{n} \cdot \frac{n\tau^2}{n\tau^2 + \sigma^2} \longrightarrow \frac{\sigma^2}{n} \quad \text{as } \tau^2 \rightarrow \infty.$$

Now let δ be any rule. For each τ ,

$$\sup_{\mu} R(\mu, \delta) \geq r(\tau, \delta) \geq r(\tau).$$

Letting $\tau^2 \rightarrow \infty$ gives $\sup_{\mu} R(\mu, \delta) \geq \sigma^2/n = \sup_{\mu} R(\mu, \delta_0)$. Hence no rule has smaller maximum risk than $\bar{X}$, which is therefore minimax.

Note 16.5 (Least favourable sequences)

The prior sequence $\{N(0, \tau^2) : \tau^2 \rightarrow \infty\}$ above is called a *least favourable sequence of priors*. A rule δ_0 is minimax if there exists a sequence of priors g_m with

$$r(g_m) \rightarrow \sup_{\theta} R(\theta, \delta_0).$$

This extends Theorem 14.2 to cases where no least favourable proper prior exists, which is typical when Θ is unbounded.

Practice (Lecture Notes, Chapter 1)

- Exercises (Section 1.13): 1.9, 1.13, 1.21.

Next lecture

Lecture 17: Complete, essentially complete and minimal complete classes and their relationship.