

Lecture 17: Complete, essentially complete and minimal complete classes and their relationship

Statistical Inference–II ■ Unit I: Decision Theory

Kapil Kumar

Department of Statistics
Chaudhary Charan Singh University, Meerut

M.Sc. Statistics, Semester III
Academic Session 2026–2027

Previous lecture

Lecture 16: Minimax estimators of a binomial proportion and of a normal mean.

Today

- Complete Classes of Decision Rules
 - Definitions
 - The Relationship between the Classes

Admissibility identifies rules that should not be used. It is natural to ask whether one can identify, once and for all, a manageable subclass of $\mathcal{D}$ that is guaranteed to contain a satisfactory rule. This is the purpose of complete class theory.

Definition 17.1 (Complete class)

A class $\mathcal{C} \subseteq \mathcal{D}$ of decision rules is *complete* if for every rule $\delta \notin \mathcal{C}$ there exists a rule $\delta_0 \in \mathcal{C}$ which is uniformly better than δ , that is,

$$\begin{aligned} R(\theta, \delta_0) &\leq R(\theta, \delta) \quad \text{for all } \theta \in \Theta, \\ \text{and } R(\theta, \delta_0) &< R(\theta, \delta) \quad \text{for some } \theta \in \Theta. \end{aligned}$$

Definition 17.2 (Essentially complete class)

A class $\mathcal{C} \subseteq \mathcal{D}$ is *essentially complete* if for every rule $\delta \notin \mathcal{C}$ there exists $\delta_0 \in \mathcal{C}$ with

$$R(\theta, \delta_0) \leq R(\theta, \delta) \quad \text{for all } \theta \in \Theta.$$

Remark 17.3 (The distinction)

The only difference is that essential completeness does *not* require the strict inequality at any point. Thus δ_0 need only be “as good as” δ , whereas completeness requires it to be strictly better somewhere.

Consequently

$$\text{complete} \implies \text{essentially complete},$$

but not conversely. Essential completeness is the more useful notion in practice, because it permits the discarding of rules that are merely duplicated in risk.

Definition 17.4 (Minimal complete class)

A complete class $\mathcal{C}$ is *minimal complete* if no proper subset of $\mathcal{C}$ is complete. *Minimal essentially complete* is defined analogously.

Theorem 17.5 (Structure of the minimal complete class)

If a minimal complete class $\mathcal{C}$ exists, then it consists exactly of the admissible rules; that is,

$$\mathcal{C} = \mathcal{D}_A := \{\delta \in \mathcal{D} : \delta \text{ is admissible}\}.$$

Proof.

(i) Every admissible rule belongs to $\mathcal{C}$. Let δ be admissible and suppose $\delta \notin \mathcal{C}$. Since $\mathcal{C}$ is complete, there is $\delta_0 \in \mathcal{C}$ uniformly better than δ ; but then δ is inadmissible, a contradiction. Hence $\mathcal{D}_A \subseteq \mathcal{C}$.

Proof (cont.)

(ii) Every member of $\mathcal{C}$ is admissible. Suppose some $\delta_1 \in \mathcal{C}$ is inadmissible. We show that $\mathcal{C} \setminus \{\delta_1\}$ is still complete, contradicting minimality.

Let $\delta \notin \mathcal{C} \setminus \{\delta_1\}$. Two cases arise.

- If $\delta = \delta_1$: since δ_1 is inadmissible, some $\delta' \in \mathcal{D}$ is uniformly better; note that $\delta' \neq \delta_1$, since δ' is strictly better than δ_1 somewhere. If $\delta' \in \mathcal{C}$ we are done. Otherwise $\delta' \notin \mathcal{C}$; apply completeness of $\mathcal{C}$ to δ' to obtain a member of $\mathcal{C}$ uniformly better than δ' , and hence (by transitivity of domination) than δ_1 ; this member differs from δ_1 since a rule cannot be uniformly better than itself.

Proof (cont.)

- If $\delta \notin \mathcal{C}$: by completeness of $\mathcal{C}$ there is $\delta_0 \in \mathcal{C}$ uniformly better than δ . If $\delta_0 \neq \delta_1$ we are done. If $\delta_0 = \delta_1$, then by the previous case there is $\delta'' \in \mathcal{C} \setminus \{\delta_1\}$ uniformly better than δ_1 , and therefore uniformly better than δ .

In either case $\mathcal{C} \setminus \{\delta_1\}$ is complete, contradicting the minimality of $\mathcal{C}$.
Hence $\mathcal{C} \subseteq \mathcal{D}_A$.

Combining (i) and (ii), $\mathcal{C} = \mathcal{D}_A$. □

Corollary 17.6

A minimal complete class, if it exists, is unique.

Proof.

Immediate from Theorem 17.5, since the set of admissible rules is determined by the problem alone. □

Theorem 17.7 (Admissible rules and completeness)

If the class $\mathcal{D}_A$ of all admissible rules is complete, then it is the minimal complete class.

Proof.

Any complete class must, by part (i) of the proof above, contain $\mathcal{D}_A$. Hence if $\mathcal{D}_A$ is itself complete, no proper subset of it can be complete, so it is minimal. □

Remark 17.8 (Summary of the relationships)

For the syllabus, the following chain should be memorised:

(R1) Every complete class is essentially complete.

(R2) A minimal complete class, when it exists, equals the class of all admissible rules, and is unique.

(R3) A minimal essentially complete class need not be unique (rules with identical risk functions may be interchanged freely).

(R4) A complete class must contain every admissible rule.

(R5) If a minimal complete class exists and δ is not in it, then δ is inadmissible and should not be used.

Practice (Lecture Notes, Chapter 1)

- Short-answer questions (Section 1.12): Q13, Q14.

Next lecture

Lecture 18: Complete class theorems; sufficiency; estimation viewed as a decision problem.