

Lecture 19: Testing viewed as a decision problem; the Bayes test

Statistical Inference–II ■ Unit I: Decision Theory

Kapil Kumar

Department of Statistics
Chaudhary Charan Singh University, Meerut

M.Sc. Statistics, Semester III
Academic Session 2026–2027

Previous lecture

Lecture 18: Complete class theorems; sufficiency; estimation viewed as a decision problem.

Today

- Testing Viewed as a Decision Problem
 - Set-up
 - The Bayes Test in the Simple versus Simple Case

We now develop the syllabus items “estimation and testing viewed as a decision rule problem” and “Bayes and minimax tests in simple cases”.

Let $\Theta = \Theta_0 \cup \Theta_1$ be a partition and let $\mathcal{A} = \{a_0, a_1\}$, where a_0 means “do not reject $H_0 : \theta \in \Theta_0$ ” and a_1 means “reject H_0 ”. A randomised decision rule is described by its test function

$$\phi(\mathbf{x}) = \mathbb{P}(\text{take } a_1 \mid \mathbf{X} = \mathbf{x}) \in [0, 1].$$

The Risk of a Test

Under the testing loss function introduced earlier, and using the example “Risk of a test under the testing loss (c_{01}, c_{10}) ”,

$$R(\theta, \phi) = \begin{cases} c_{01} \mathbb{E}_{\theta}[\phi(\mathbf{X})], & \theta \in \Theta_0, \\ c_{10} \mathbb{E}_{\theta}[1 - \phi(\mathbf{X})], & \theta \in \Theta_1. \end{cases}$$

Thus $\mathbb{E}_{\theta}[\phi(\mathbf{X})]$ is the power function, and the risk is c_{01} times the Type I error probability $\alpha(\theta) = \mathbb{E}_{\theta}\phi(\mathbf{X})$ for $\theta \in \Theta_0$ and c_{10} times the Type II error probability on Θ_1 .

Remark 19.1 (Neyman–Pearson versus decision theory)

The Neyman–Pearson approach fixes α and minimises the Type II error — a restricted-class strategy of type (A). The decision-theoretic approach instead weights the two errors by c_{01} and c_{10} and minimises a combination. In the simple-versus-simple case the two lead to the same family of tests (likelihood-ratio tests) but select different members of it.

The Bayes Test in the Simple versus Simple Case

Theorem 19.2 (Bayes test)

Let $\Theta = \{\theta_0, \theta_1\}$, with densities $f_0(\mathbf{x}) = f(\mathbf{x} \mid \theta_0)$ and $f_1(\mathbf{x}) = f(\mathbf{x} \mid \theta_1)$, and let the prior assign probabilities $\pi_0 = g(\theta_0)$ and $\pi_1 = g(\theta_1) = 1 - \pi_0$. Under the testing loss function introduced earlier, with $L(\theta_0, a_1) = c_{01}$ and $L(\theta_1, a_0) = c_{10}$, the Bayes test rejects H_0 when

$$\frac{f_1(\mathbf{x})}{f_0(\mathbf{x})} > \frac{\pi_0 c_{01}}{\pi_1 c_{10}},$$

does not reject H_0 when the inequality is reversed, and may do anything on the boundary. That is, the Bayes test is a likelihood ratio test with critical value $k = \pi_0 c_{01} / (\pi_1 c_{10})$.

Proof.

By the Structure Theorem (structure of the Bayes rule) we minimise, for each $\mathbf{x}$, the posterior expected loss. The posterior probabilities are

$$\pi(\theta_0 \mid \mathbf{x}) = \frac{\pi_0 f_0(\mathbf{x})}{p(\mathbf{x})}, \quad \pi(\theta_1 \mid \mathbf{x}) = \frac{\pi_1 f_1(\mathbf{x})}{p(\mathbf{x})},$$

where

$$p(\mathbf{x}) = \pi_0 f_0(\mathbf{x}) + \pi_1 f_1(\mathbf{x}).$$

Proof (cont.)

The posterior expected losses of the two actions are

$$\begin{aligned}\rho(\pi, a_0 \mid \mathbf{x}) &= L(\theta_0, a_0)\pi(\theta_0 \mid \mathbf{x}) + L(\theta_1, a_0)\pi(\theta_1 \mid \mathbf{x}) \\ &= 0 + c_{10} \frac{\pi_1 f_1(\mathbf{x})}{p(\mathbf{x})},\end{aligned}$$

$$\begin{aligned}\rho(\pi, a_1 \mid \mathbf{x}) &= L(\theta_0, a_1)\pi(\theta_0 \mid \mathbf{x}) + L(\theta_1, a_1)\pi(\theta_1 \mid \mathbf{x}) \\ &= c_{01} \frac{\pi_0 f_0(\mathbf{x})}{p(\mathbf{x})} + 0.\end{aligned}$$

Proof (cont.)

Action a_1 (reject) is preferred when $\rho(\pi, a_1 \mid \mathbf{x}) < \rho(\pi, a_0 \mid \mathbf{x})$, i.e. when

$$c_{01} \pi_0 f_0(\mathbf{x}) < c_{10} \pi_1 f_1(\mathbf{x}),$$

after multiplying through by $p(\mathbf{x}) > 0$. Rearranging,

$$\frac{f_1(\mathbf{x})}{f_0(\mathbf{x})} > \frac{\pi_0 c_{01}}{\pi_1 c_{10}},$$

as claimed. The two actions are equally good when equality holds, so the behaviour on the boundary is immaterial. □

Corollary 19.3

The minimum Bayes risk is

$$r(g) = \int_{\mathcal{X}} \min \{c_{10}\pi_1 f_1(\mathbf{x}), c_{01}\pi_0 f_0(\mathbf{x})\} d\mathbf{x}.$$

Example: Bayes Test for a Normal Mean

Example 19.4 (Bayes test for a normal mean)

Let $X_1, \dots, X_n$ be i.i.d. $N(\theta, \sigma^2)$ with σ^2 known, and test $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$ where $\theta_1 > \theta_0$. Take $\pi_0 = \pi_1 = \frac{1}{2}$ and $c_{01} = c_{10} = 1$ (0-1 loss).

Solution

The likelihood ratio is

$$\Lambda(\mathbf{x}) = \frac{f_1(\mathbf{x})}{f_0(\mathbf{x})} = \exp \left\{ -\frac{1}{2\sigma^2} \left[\sum_i (x_i - \theta_1)^2 - \sum_i (x_i - \theta_0)^2 \right] \right\}.$$

Solution (cont.)

Expanding the bracket,

$$\sum_i (x_i - \theta_1)^2 - \sum_i (x_i - \theta_0)^2 = -2(\theta_1 - \theta_0) \sum_i x_i + n(\theta_1^2 - \theta_0^2),$$

so

$$\Lambda(\mathbf{x}) = \exp \left\{ \frac{n(\theta_1 - \theta_0)}{\sigma^2} \bar{x} - \frac{n(\theta_1^2 - \theta_0^2)}{2\sigma^2} \right\}.$$

Solution (cont.)

By Theorem 19.2 with $k = \pi_0 c_{01} / (\pi_1 c_{10}) = 1$, we reject H_0 when $\Lambda(\mathbf{x}) > 1$, i.e. when

$$\frac{n(\theta_1 - \theta_0)}{\sigma^2} \bar{x} > \frac{n(\theta_1^2 - \theta_0^2)}{2\sigma^2}.$$

Since $\theta_1 > \theta_0$ we may divide by the positive quantity $n(\theta_1 - \theta_0)/\sigma^2$, obtaining

$$\text{Reject } H_0 \iff \bar{x} > \frac{\theta_0 + \theta_1}{2}$$

The Bayes test with equal priors and equal losses simply compares the sample mean with the midpoint of the two hypothesised values — an intuitively compelling rule.

Solution (cont.)

Its Bayes risk is

$$\begin{aligned} r(g) &= \frac{1}{2} \mathbb{P}_{\theta_0} \left(\bar{X} > \frac{\theta_0 + \theta_1}{2} \right) + \frac{1}{2} \mathbb{P}_{\theta_1} \left(\bar{X} \leq \frac{\theta_0 + \theta_1}{2} \right) \\ &= \Phi \left(-\frac{\sqrt{n}(\theta_1 - \theta_0)}{2\sigma} \right), \end{aligned}$$

using the symmetry of the normal distribution, where Φ is the standard normal distribution function.

Practice (Lecture Notes, Chapter 1)

- Short-answer questions (Section 1.12): Q16.

Next lecture

Lecture 20: The minimax test; summary and revision.