

Lecture 20: The minimax test; summary and revision

Statistical Inference–II ■ Unit I: Decision Theory

Kapil Kumar

Department of Statistics
Chaudhary Charan Singh University, Meerut

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Previous lecture

Lecture 19: Testing viewed as a decision problem; the Bayes test.

Today

- Testing Viewed as a Decision Problem
 - The Minimax Test in the Simple versus Simple Case
- Summary and Formula Sheet
- Short-Answer Questions

The Minimax Test in the Simple versus Simple Case

Theorem 20.1 (Minimax test)

In the setting of Theorem 19.2, the minimax test is the likelihood ratio test whose critical value k is chosen so that the two risks are equal:

$$c_{01} \alpha(k) = c_{10} \beta(k),$$

where $\alpha(k) = \mathbb{P}_{\theta_0}(\Lambda(\mathbf{X}) > k)$ and $\beta(k) = \mathbb{P}_{\theta_1}(\Lambda(\mathbf{X}) \leq k)$.

Randomisation may be required to achieve exact equality when the distribution of Λ is discrete.

Proof.

Since Θ has two points, the risk of any rule ϕ is the pair $(c_{01}\alpha(\phi), c_{10}\beta(\phi))$. The test ϕ_k described has equal risks, so it is an equaliser rule.

Proof (cont.)

Choose

$$\pi_0 = \frac{c_{10}k}{c_{10}k + c_{01}} \quad \text{so that} \quad k = \pi_0 c_{01} / (\pi_1 c_{10});$$

then by Theorem 19.2, ϕ_k is the Bayes test with respect to this prior. Being Bayes with constant risk, it is minimax by Corollary 14.3, and the corresponding prior is least favourable. □

Example 20.2 (Minimax test for a normal mean)

In the setting of Example 19.4 with $c_{01} = c_{10} = 1$, the LRT with cut-off c has

$$\alpha(c) = \mathbb{P}_{\theta_0}(\bar{X} > c) = 1 - \Phi\left(\frac{\sqrt{n}(c - \theta_0)}{\sigma}\right),$$
$$\beta(c) = \Phi\left(\frac{\sqrt{n}(c - \theta_1)}{\sigma}\right).$$

Example 20.2 (cont.)

Setting $\alpha(c) = \beta(c)$ and using $1 - \Phi(u) = \Phi(-u)$:

$$\begin{aligned}\Phi\left(-\frac{\sqrt{n}(c - \theta_0)}{\sigma}\right) &= \Phi\left(\frac{\sqrt{n}(c - \theta_1)}{\sigma}\right) \\ \implies -(c - \theta_0) &= c - \theta_1 \implies c = \frac{\theta_0 + \theta_1}{2}.\end{aligned}$$

Hence the minimax test coincides with the Bayes test under the uniform prior — confirming that $\pi_0 = \pi_1 = \frac{1}{2}$ is least favourable here, as symmetry suggests. The minimax risk is $\Phi(-\sqrt{n}(\theta_1 - \theta_0)/(2\sigma))$.

Example 20.3 (A discrete minimax test requiring randomisation)

Let $X \sim \text{Bernoulli}(p)$ with one observation, and test

$$H_0 : p = \frac{1}{4} \quad \text{against} \quad H_1 : p = \frac{1}{2}$$

under equal losses $c_{01} = c_{10} = 1$. The likelihood ratios are

$$\Lambda(0) = \frac{1 - 1/2}{1 - 1/4} = \frac{2}{3}, \quad \Lambda(1) = \frac{1/2}{1/4} = 2.$$

Example 20.3 (cont.)

Among non-randomised likelihood-ratio tests, the best rule rejects only when $X = 1$. Its risks are

$$\alpha = \mathbb{P}_{1/4}(X = 1) = \frac{1}{4}, \quad \beta = \mathbb{P}_{1/2}(X = 0) = \frac{1}{2},$$

so its maximum risk is $1/2$.

Now reject whenever $X = 1$ and, when $X = 0$, reject with probability q . Then

$$\alpha(q) = \frac{1}{4} + \frac{3}{4}q, \quad \beta(q) = \frac{1}{2}(1 - q).$$

Example 20.3 (cont.)

The minimax equaliser condition gives

$$\frac{1}{4} + \frac{3}{4}q = \frac{1}{2}(1 - q) \implies q = \frac{1}{5}.$$

Therefore the randomised test

$$\phi(1) = 1, \quad \phi(0) = \frac{1}{5}$$

has constant risk

$$\alpha = \beta = \frac{2}{5} < \frac{1}{2},$$

and is minimax.

Example 20.3 (cont.)

Since the critical likelihood ratio is $k = 2/3$, the least favourable prior satisfies $\pi_0/\pi_1 = 2/3$, namely

$$\pi_0 = \frac{2}{5}, \quad \pi_1 = \frac{3}{5}.$$

This example shows concretely why randomisation may be necessary in a discrete testing problem.

1. A statistical decision problem is $(\Theta, \mathcal{A}, L, \{F_\theta\})$. A rule $\delta : \mathcal{X} \rightarrow \mathcal{A}$ is judged by its risk function $R(\theta, \delta) = \mathbb{E}_\theta L(\theta, \delta(\mathbf{X}))$.
2. Risk functions are *functions*; a uniformly best rule almost never exists. Three ways out: restrict the class (unbiasedness), average the risk (Bayes), or take its supremum (minimax).
3. Admissibility (no dominating rule) is a minimal requirement only.
4. The Bayes rule is found pointwise: for each $\mathbf{x}$, minimise the posterior expected loss.
5. A Bayes rule that is unique is admissible. A Bayes rule with constant risk is minimax, and its prior is least favourable.
6. Randomisation can lower the minimax risk, but never helps under a convex loss.
7. A minimal complete class, if it exists, is the class of admissible rules.
8. The Bayes test and the minimax test for simple hypotheses are both likelihood ratio tests; they differ only in the critical value.

Quantity	Formula
Risk	$R(\theta, \delta) = \mathbb{E}_\theta L(\theta, \delta(\mathbf{X}))$; under SELF, $R = \text{Var}_\theta(\delta) + \{\text{bias}(\theta)\}^2$
Risk of a test	$c_{01} \alpha(\theta)$ on Θ_0 ; $c_{10} \beta(\theta)$ on Θ_1
Mixture of two rules	$R(\theta, \lambda\delta_1 + (1 - \lambda)\delta_2) = \lambda R(\theta, \delta_1) + (1 - \lambda)R(\theta, \delta_2)$
Minimax rule	minimises $\sup_\theta R(\theta, \delta)$
Bayes risk	$r(g, \delta) = \int_\Theta R(\theta, \delta)g(\theta) d\theta$
Posterior	$\pi(\theta \mathbf{x}) = f(\mathbf{x} \theta)g(\theta)/p(\mathbf{x})$, $p(\mathbf{x}) = \int f(\mathbf{x} \theta)g(\theta) d\theta$

Quantity	Formula
Bayes estimators	SELF: $\mathbb{E}(\theta \mid \mathbf{x})$; weighted SELF: $\mathbb{E}\{w(\theta)\theta \mid \mathbf{x}\}/\mathbb{E}\{w(\theta) \mid \mathbf{x}\}$; ALF: posterior median; small-neighbourhood 0–1: $\approx$ posterior mode; LINEX: $-c^{-1} \log \mathbb{E}(e^{-c\theta} \mid \mathbf{x})$
Bayes risk under SELF	$r(g) = \mathbb{E}\{\text{Var}(\theta \mid \mathbf{X})\}$
Beta–binomial	prior $\text{Beta}(\alpha, \beta)$: posterior $\text{Beta}(x + \alpha, n - x + \beta)$, $\hat{p} = (x + \alpha)/(n + \alpha + \beta)$; uniform prior: $r(g) = 1/\{6(n + 2)\}$
Normal–normal	$N(\mu, 1)$ data, prior $N(0, \tau^2)$: $\hat{\mu} = n\tau^2\bar{x}/(n\tau^2 + 1)$, $r(g) = \tau^2/(n\tau^2 + 1)$

Quantity	Formula
Poisson–gamma	prior $\text{Gamma}(\alpha, \beta)$: $\hat{\theta} = (S + \alpha)/(n + \beta)$, $r(g) = \alpha/\{\beta(n + \beta)\}$
Least favourable prior	if $r(g, \delta_g) \geq \sup_{\theta} R(\theta, \delta_g)$ then δ_g is minimax and g least favourable
Minimax estimators	binomial: $\delta_M = (X + \frac{1}{2}\sqrt{n})/(n + \sqrt{n})$, risk $1/\{4(1 + \sqrt{n})^2\}$; normal mean: $\bar{X}$
Classes of rules	complete $\Rightarrow$ essentially complete; minimal complete (if it exists) = admissible rules
Bayes test	reject H_0 if $f_1(\mathbf{x})/f_0(\mathbf{x}) > \pi_0 c_{01}/(\pi_1 c_{10})$
Minimax test	likelihood ratio test with $c_{01} \alpha = c_{10} \beta$

- Q1. Define the loss function and the risk function. Why is the risk, and not the loss, used to compare decision rules?
- Q2. Distinguish between a randomised and a non-randomised decision rule.
- Q3. Show that under squared error loss the risk of an estimator is its mean squared error.
- Q4. Write the risk function of a test under the testing loss (c_{01}, c_{10}) .
- Q5. When is a decision rule admissible? Give an example of an admissible rule that is nevertheless unreasonable.
- Q6. Why does a uniformly best estimator rarely exist?
- Q7. Define a minimax rule. Why is the minimax principle called pessimistic?
- Q8. Define the Bayes risk and a Bayes rule.

Short-answer questions (cont.)

- Q9. Distinguish between the posterior expected loss and the Bayes risk.
- Q10. State the Bayes estimator under squared error, absolute error and LINEX loss.
- Q11. What is a least favourable prior? What is an equaliser rule?
- Q12. State a sufficient condition for a Bayes rule to be minimax.
- Q13. Define complete, essentially complete and minimal complete classes of decision rules.
- Q14. State the relationship between the minimal complete class and the class of admissible rules.
- Q15. Why is randomisation of no use in estimation under squared error loss?
- Q16. State the form of the Bayes test for testing a simple hypothesis against a simple alternative.

Practice (Lecture Notes, Chapter 1)

- Short-answer questions (Section 1.12): Q1–Q16 (all, for revision).
- Exercises (Section 1.13): 1.7.

Next lecture

Lecture 21 begins Units II–III: Sequential Analysis.