

Lecture 22: Wald's SPRT; the log-likelihood ratio as a random walk

Statistical Inference-II ■ Units II–III: Sequential Analysis

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Previous lecture

Lecture 21: Sequential procedures; stopping rules and stopping times.

Today

- Sequential Probability Ratio Test

Two simple hypotheses and their likelihoods

Suppose $X_1, X_2, \dots$ are i.i.d. with p.m.f. or p.d.f. $f(x; \theta)$. Consider the two simple hypotheses

$$H_0 : \theta = \theta_0 \quad \text{against} \quad H_1 : \theta = \theta_1.$$

Write $f_j(x) = f(x; \theta_j)$, $j = 0, 1$, and assume that f_0 and f_1 have the same support and are not identical. After m observations define the likelihoods

$$\mathcal{L}_{j,m} = \prod_{i=1}^m f(X_i; \theta_j), \quad j = 0, 1.$$

The likelihood ratio after m observations is

$$\Lambda_m = \frac{\mathcal{L}_{1,m}}{\mathcal{L}_{0,m}} = \prod_{i=1}^m \frac{f(X_i; \theta_1)}{f(X_i; \theta_0)}.$$

For numerical work it is convenient to use the log-likelihood-ratio increments

$$Z_i = \log \frac{f(X_i; \theta_1)}{f(X_i; \theta_0)}, \quad W_m = \sum_{i=1}^m Z_i = \log \Lambda_m.$$

Under any fixed θ the Z_i are i.i.d., so W_m is a random walk.

Wald's sequential probability ratio test

Choose constants A and B satisfying $0 < B < 1 < A$.

Definition 22.1 (SPRT)

Wald's sequential probability ratio test (SPRT) is

$$\begin{aligned}\Lambda_m \leq B &\iff W_m \leq \log B : \text{ stop and accept } H_0, \\ B < \Lambda_m < A &\iff \log B < W_m < \log A : \text{ continue sampling,} \\ \Lambda_m \geq A &\iff W_m \geq \log A : \text{ stop and reject } H_0.\end{aligned}$$

The stopping time is therefore

$$N = \inf\{m \geq 1 : W_m \notin (\log B, \log A)\}.$$

It is shown in the next section, on termination of the SPRT, that $\mathbb{P}_\theta(N < \infty) = 1$: the test terminates with probability one.

A sample path of the SPRT

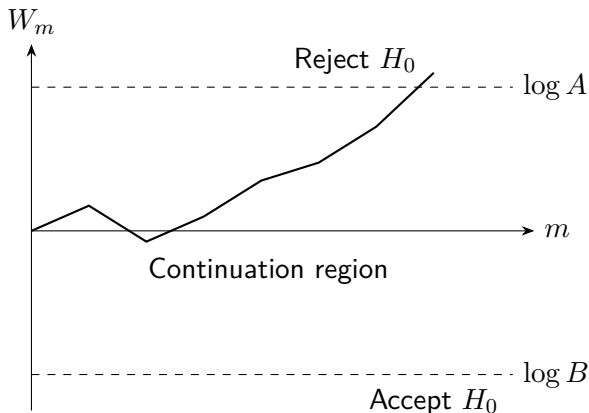


Figure 22.1: A sample path of the cumulative log-likelihood ratio in an SPRT.

Practice (Lecture Notes, Chapter 2)

- Short-answer questions (Section 2.12): Q3.

Next lecture

Lecture 23: Termination of the SPRT with probability one.