

Lecture 23: Termination of the SPRT with probability one

Statistical Inference–II ■ Units II–III: Sequential Analysis

Kapil Kumar

Department of Statistics
Chaudhary Charan Singh University, Meerut

M.Sc. Statistics, Semester III
Academic Session 2026–2027

Previous lecture

Lecture 22: Wald's SPRT; the log-likelihood ratio as a random walk.

Today

- Termination of the SPRT with Probability One

Theorem 23.1 (Almost-sure termination)

Let $W_m = \sum_{i=1}^m Z_i$, where the increments are i.i.d. under $\mathbb{P}_\theta$, and suppose $\mathbb{P}_\theta(Z_1 = 0) < 1$. For finite boundaries $-\infty < \log B < 0 < \log A < \infty$, the SPRT stopping time

$$N = \inf\{m \geq 1 : W_m \notin (\log B, \log A)\}$$

is finite with probability one. In fact, there exist an integer $k \geq 1$ and a constant $0 < \rho < 1$ such that

$$\mathbb{P}_\theta(N > n) \leq \rho^{\lfloor n/k \rfloor}, \quad n \geq 0.$$

Consequently, $\mathbb{E}_\theta(N^r) < \infty$ for every positive integer r .

Proof.

Since Z_1 is not identically zero, at least one of $\mathbb{P}_\theta(Z_1 > 0)$ and $\mathbb{P}_\theta(Z_1 < 0)$ is positive. Suppose first that $\mathbb{P}_\theta(Z_1 > 0) > 0$. Then there exists $c > 0$ such that $\mathbb{P}_\theta(Z_1 \geq c) = q > 0$. Choose k so large that $kc > \log A - \log B$. From any point in the continuation interval, k consecutive increments each at least c force the path to cross the upper boundary within the next k observations. Therefore, conditionally on all information available at the start of a block,

$$\mathbb{P}_\theta(\text{no termination during the next } k \text{ observations} \mid \mathcal{F}_m) \leq 1 - q^k.$$

Proof (cont.)

The case $\mathbb{P}_\theta(Z_1 < 0) > 0$ is analogous, using k consecutive increments not exceeding a negative constant and crossing the lower boundary. Thus in either case there is an $\eta > 0$ such that the conditional probability of termination within any block of k further observations is at least η . By repeated conditioning,

$$\mathbb{P}_\theta(N > jk) \leq (1 - \eta)^j.$$

Taking $\rho = 1 - \eta$ yields the stated geometric bound. Letting $j \rightarrow \infty$ gives $\mathbb{P}_\theta(N = \infty) = 0$. Finally,

$$\mathbb{E}_\theta(N^r) = \sum_{n \geq 0} \{(n+1)^r - n^r\} \mathbb{P}_\theta(N > n) \leq \sum_{n \geq 0} (n+1)^r \rho^{\lfloor n/k \rfloor} < \infty,$$

because a polynomial times a geometric sequence is summable. □

Remark 23.2

The condition $\mathbb{P}_\theta(Z_1 = 0) < 1$ excludes the degenerate situation in which every observation produces likelihood ratio one and the data can never move the process towards either boundary. Under the standing assumption of the SPRT section (f_0 and f_1 have the same support and are not identical) this degeneracy cannot occur under θ_0 or θ_1 : if, say, $\mathbb{P}_{\theta_0}(Z_1 = 0) = 1$, then $f_1(X) = f_0(X)$ almost surely under f_0 , so $f_1 = f_0$ almost everywhere on the common support, i.e. the two densities coincide.

Practice (Lecture Notes, Chapter 2)

- Short-answer questions (Section 2.12): Q7.
- Exercises (Section 2.13): 2.8.

Next lecture

Lecture 24: Choice of the boundaries; Wald's approximations.