

Lecture 24: Choice of the boundaries; Wald's approximations

Statistical Inference–II ■ Units II–III: Sequential Analysis

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Previous lecture

Lecture 23: Termination of the SPRT with probability one.

Today

- Boundaries and Error Probabilities of the SPRT
 - Choice of the boundaries

The constants A and B are chosen to give prescribed error probabilities. Throughout this section $\mathbb{P}_\theta(N < \infty) = 1$, by the almost-sure termination theorem.

Let

$$\alpha' = \mathbb{P}_{\theta_0}(\text{reject } H_0), \quad \beta' = \mathbb{P}_{\theta_1}(\text{accept } H_0)$$

denote the error probabilities actually attained by the SPRT with boundaries (A, B) ; the unprimed α and β are reserved for the nominal (desired) error probabilities.

For $n \geq 1$ let R_n^a (respectively R_n^r) be the set of points $(x_1, \dots, x_n)$ for which the test continues at stages $1, \dots, n-1$ and accepts (respectively rejects) H_0 at stage n , and write $d\mathbf{x}_n = dx_1 \cdots dx_n$.

The event “accept H_0 ” is the union over $n \geq 1$ of the disjoint events $\{(X_1, \dots, X_n) \in R_n^a\}$, and on R_n^a we have $\Lambda_n \leq B$, that is, $\prod_{i=1}^n f_1(x_i) \leq B \prod_{i=1}^n f_0(x_i)$. Summing over the stages,

$$\begin{aligned}\beta' &= \sum_{n=1}^{\infty} \int_{R_n^a} \prod_{i=1}^n f_1(x_i) d\mathbf{x}_n \\ &\leq B \sum_{n=1}^{\infty} \int_{R_n^a} \prod_{i=1}^n f_0(x_i) d\mathbf{x}_n = B(1 - \alpha'),\end{aligned}$$

with sums in place of integrals in the discrete case; the last step uses $\mathbb{P}_{\theta_0}(\text{accept } H_0) = 1 - \alpha'$, which holds because $\mathbb{P}_{\theta_0}(N < \infty) = 1$ (almost-sure termination theorem).

Similarly, on R_n^r we have $\Lambda_n \geq A$, and hence, using $\mathbb{P}_{\theta_1}(N < \infty) = 1$,

$$\begin{aligned} 1 - \beta' &= \sum_{n=1}^{\infty} \int_{R_n^r} \prod_{i=1}^n f_1(x_i) d\mathbf{x}_n \\ &\geq A \sum_{n=1}^{\infty} \int_{R_n^r} \prod_{i=1}^n f_0(x_i) d\mathbf{x}_n = A\alpha'. \end{aligned}$$

Therefore

$$B \geq \frac{\beta'}{1 - \alpha'}, \quad A \leq \frac{1 - \beta'}{\alpha'}. \quad (24.1)$$

Because the likelihood ratio generally overshoots the boundaries, the attained error probabilities need not equal the nominal values. Wald's standard working approximations, obtained by treating (24.1) as equalities with the attained α', β' replaced by the nominal α, β , are

$$A \approx \frac{1 - \beta}{\alpha}, \quad B \approx \frac{\beta}{1 - \alpha}.$$

Practice (Lecture Notes, Chapter 2)

- Short-answer questions (Section 2.12): Q4.

Next lecture

Lecture 25: Wald's bounds on the attained error probabilities; the SPRT in terms of a sufficient statistic.