

Lecture 25: Wald's bounds on the attained error probabilities; the SPRT in terms of a sufficient statistic

Statistical Inference-II ■ Units II–III: Sequential Analysis

Kapil Kumar

Department of Statistics
Chaudhary Charan Singh University, Meerut

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Previous lecture

Lecture 24: Choice of the boundaries; Wald's approximations.

Today

- Boundaries and Error Probabilities of the SPRT
 - Wald's bounds on the attained error probabilities
 - The SPRT in terms of a sufficient statistic

The next result shows that using Wald's approximations is safe: the attained error probabilities can exceed the nominal ones only slightly, and their sum cannot exceed the nominal sum.

Theorem 25.1 (Wald's bounds)

Let $0 < \alpha, \beta < 1$ with $\alpha + \beta < 1$ be the desired error probabilities, and let the SPRT use

$$A = \frac{1 - \beta}{\alpha}, \quad B = \frac{\beta}{1 - \alpha}.$$

If α' and β' denote the error probabilities actually attained by this test, then

$$\alpha' \leq \frac{\alpha}{1 - \beta}, \quad \beta' \leq \frac{\beta}{1 - \alpha}, \quad \alpha' + \beta' \leq \alpha + \beta.$$

Proof.

The inequalities (24.1) hold for the attained error probabilities α', β' of *any* SPRT. Applied to the present test they read

$$\frac{\beta'}{1 - \alpha'} \leq B = \frac{\beta}{1 - \alpha}, \quad \frac{1 - \beta'}{\alpha'} \geq A = \frac{1 - \beta}{\alpha}.$$

Cross-multiplying,

$$\beta'(1 - \alpha) \leq \beta(1 - \alpha'), \quad \alpha'(1 - \beta) \leq \alpha(1 - \beta'). \quad (25.1)$$

Proof (cont.)

The second inequality in (25.1) gives

$$\alpha' \leq \alpha(1 - \beta')/(1 - \beta) \leq \alpha/(1 - \beta),$$

and the first gives

$$\beta' \leq \beta(1 - \alpha')/(1 - \alpha) \leq \beta/(1 - \alpha).$$

Adding the two inequalities in (25.1),

$$\beta' - \alpha\beta' + \alpha' - \alpha'\beta \leq \beta - \alpha'\beta + \alpha - \alpha\beta',$$

and cancelling the common terms $-\alpha\beta'$ and $-\alpha'\beta$ gives

$$\alpha' + \beta' \leq \alpha + \beta.$$



Remark 25.2

For $\alpha = 0.05$ and $\beta = 0.10$ the bounds are $\alpha' \leq 0.0556$, $\beta' \leq 0.1053$ and $\alpha' + \beta' \leq 0.15$. At most one of the two attained error probabilities can exceed its nominal value, and then only by a small amount. In practice both are usually *smaller* than nominal because of overshoot; see the complete worked example on O.C. and A.S.N. curves later in this chapter.

In all the models of the section on distribution-specific SPRTs the density has the one-parameter exponential-family form

$$f(x; \theta) = c(\theta) k(x) \exp\{q(\theta) T(x)\}.$$

Then, writing $S_m = \sum_{i=1}^m T(X_i)$,

$$W_m = m \log \frac{c(\theta_1)}{c(\theta_0)} + \{q(\theta_1) - q(\theta_0)\} S_m,$$

which is linear in both m and S_m .

The continuation region in terms of S_m

If $q(\theta_1) > q(\theta_0)$, the condition $\log B < W_m < \log A$ is equivalent to

$$\begin{aligned} a_m &< S_m < r_m, \quad \text{where} \\ a_m &= \frac{\log B - m \log\{c(\theta_1)/c(\theta_0)\}}{q(\theta_1) - q(\theta_0)}, \\ r_m &= \frac{\log A - m \log\{c(\theta_1)/c(\theta_0)\}}{q(\theta_1) - q(\theta_0)}. \end{aligned}$$

Sampling continues while S_m lies strictly between the *acceptance line* a_m and the *rejection line* r_m ; H_0 is accepted as soon as $S_m \leq a_m$ and rejected as soon as $S_m \geq r_m$.

The two lines are parallel with common slope

$$-\log\{c(\theta_1)/c(\theta_0)\}/\{q(\theta_1) - q(\theta_0)\}.$$

If $q(\theta_1) < q(\theta_0)$ the inequalities are reversed: H_0 is accepted for large and rejected for small values of S_m .

This form is the one used in practice, because only the running total S_m needs to be updated.

Practice (Lecture Notes, Chapter 2)

- Short-answer questions (Section 2.12): Q5, Q6.
- Exercises (Section 2.13): 2.13.

Next lecture

Lecture 26: Wald's fundamental identity for bounded stopping times.