

Lecture 26: Wald's fundamental identity for bounded stopping times

Statistical Inference-II ■ Units II–III: Sequential Analysis

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Previous lecture

Lecture 25: Wald's bounds on the attained error probabilities; the SPRT in terms of a sufficient statistic.

Today

- Wald's Fundamental Identity and Its Uses

Wald's fundamental identity: bounded stopping times

Let $Z_1, Z_2, \dots$ be i.i.d., put $W_m = \sum_{i=1}^m Z_i$, and define

$$\phi(t) = \mathbb{E}(e^{tZ_1})$$

for those t for which the expectation exists.

Theorem 26.1 (Wald's fundamental identity: bounded stopping times)

Let N be a stopping time with respect to $\mathcal{F}_m = \sigma(Z_1, \dots, Z_m)$. If N is bounded almost surely and $\phi(t) < \infty$, then

$$\mathbb{E}\left[e^{tW_N} \phi(t)^{-N}\right] = 1.$$

Proof.

Define

$$M_m(t) = e^{tW_m} \phi(t)^{-m}, \quad M_0(t) = 1,$$

so that $M_{m+1}(t) = M_m(t) e^{tZ_{m+1}}/\phi(t)$, and $M_m(t) \geq 0$ is $\mathcal{F}_m$ -measurable with $\mathbb{E}\{M_m(t)\} = 1$ (a product of m independent factors, each of expectation $\mathbb{E}(e^{tZ_i})/\phi(t) = 1$). Suppose $N \leq K$ almost surely.

Telescoping over the stages before stopping,

$$M_N(t) = M_0(t) + \sum_{m=0}^{K-1} \{M_{m+1}(t) - M_m(t)\} \mathbf{1}(N > m),$$

because the indicator keeps exactly the terms with $m = 0, 1, \dots, N - 1$.

Proof (cont.)

Each term of the sum equals

$$\mathbf{1}(N > m) M_m(t) \left\{ \frac{e^{tZ_{m+1}}}{\phi(t)} - 1 \right\}.$$

Here $\mathbf{1}(N > m) M_m(t)$ is $\mathcal{F}_m$ -measurable, because $\{N > m\} = \{N \leq m\}^c \in \mathcal{F}_m$, and it is integrable; while $e^{tZ_{m+1}}/\phi(t) - 1$ is independent of $\mathcal{F}_m$, integrable, and has expectation $\mathbb{E}(e^{tZ_{m+1}})/\phi(t) - 1 = 0$. The expectation of a product of two independent integrable random variables is the product of their expectations, so every term has expectation zero. Hence

$$\mathbb{E}\{M_N(t)\} = \mathbb{E}\{M_0(t)\} = 1,$$

which is the stated identity. □

Remark 26.2

In the language of martingales, $\{M_m(t)\}$ is a martingale with respect to $(\mathcal{F}_m)$, since

$$\mathbb{E}\{M_{m+1}(t) \mid \mathcal{F}_m\} = M_m(t) \mathbb{E}\{e^{tZ_{m+1}}\phi(t)^{-1}\} = M_m(t),$$

and the proof above is the optional stopping theorem for bounded stopping times, written out directly for this case.

Practice (Lecture Notes, Chapter 2)

- Exercises (Section 2.13): 2.9 (first part).

Next lecture

Lecture 27: The fundamental identity for the SPRT; Wald's equation.