

Lecture 27: The fundamental identity for the SPRT; Wald's equation

Statistical Inference-II ■ Units II–III: Sequential Analysis

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Previous lecture

Lecture 26: Wald's fundamental identity for bounded stopping times.

Today

- Wald's Fundamental Identity and Its Uses
 - Wald's equation

Wald's fundamental identity for the SPRT

The SPRT stopping time is *not* bounded, so the bounded-case theorem cannot be applied to it directly. The following theorem closes this gap, using the almost-sure termination theorem.

Theorem 27.1 (Wald's fundamental identity for the SPRT)

Let N be the stopping time of an SPRT with finite boundaries $\log B < 0 < \log A$, and suppose $\mathbb{P}(Z_1 = 0) < 1$. Then

$$\mathbb{E}\left[e^{tW_N}\phi(t)^{-N}\right] = 1$$

for every real t with $\phi(t) < \infty$ and $\phi(t) \geq 1$. In particular, if $h \neq 0$ satisfies $\phi(h) = 1$, then $\mathbb{E}\{e^{hW_N}\} = 1$ exactly.

Idea of the proof.

Apply the bounded-case theorem to the bounded stopping time $N \wedge n = \min(N, n)$ and let $n \rightarrow \infty$. While sampling continues, W_n stays between $\log B$ and $\log A$, so e^{tW_n} is bounded by a constant; and $\phi(t) \geq 1$ gives $\phi(t)^{-n} \leq 1$. Hence the contribution of the event $\{N > n\}$ is at most a constant multiple of $\mathbb{P}(N > n)$, which tends to zero by the almost-sure termination theorem.

(Full proof: Lecture Notes, Appendix A.)



Theorem 27.2 (Wald's equation)

Let $Z_1, Z_2, \dots$ be i.i.d. with $\mathbb{E}|Z_1| < \infty$, and let N be a stopping time with respect to a filtration $(\mathcal{F}_m)$ such that Z_m is $\mathcal{F}_m$ -measurable and Z_{m+1} is independent of $\mathcal{F}_m$ (e.g. $\mathcal{F}_m = \sigma(X_1, \dots, X_m)$ when Z_i is a function of X_i), with $\mathbb{E}(N) < \infty$. Then

$$\mathbb{E}(W_N) = \mathbb{E}(N)\mathbb{E}(Z_1).$$

Proof.

Write $W_N = \sum_{i=1}^{\infty} Z_i \mathbf{1}(N \geq i)$. The event $\{N \geq i\} \in \mathcal{F}_{i-1}$ is independent of Z_i , so

$$\begin{aligned}\mathbb{E} \sum_{i=1}^{\infty} |Z_i| \mathbf{1}(N \geq i) &= \sum_{i=1}^{\infty} \mathbb{E}|Z_i| \mathbb{P}(N \geq i) \\ &= \mathbb{E}|Z_1| \sum_{i=1}^{\infty} \mathbb{P}(N \geq i) = \mathbb{E}|Z_1| \mathbb{E}(N) < \infty.\end{aligned}$$

This justifies interchanging expectation and summation (Fubini's theorem):

$$\mathbb{E}(W_N) = \sum_{i=1}^{\infty} \mathbb{E}(Z_i) \mathbb{P}(N \geq i) = \mathbb{E}(Z_1) \sum_{i=1}^{\infty} \mathbb{P}(N \geq i) = \mathbb{E}(Z_1) \mathbb{E}(N),$$

using $\sum_{i \geq 1} \mathbb{P}(N \geq i) = \mathbb{E}(N)$ for a positive integer-valued N . □

Remark 27.3

Formally, Wald's equation also follows by differentiating $\mathbb{E}[e^{tW_N} \phi(t)^{-N}] = 1$ at $t = 0$, since $\phi(0) = 1$ and $\phi'(0) = \mathbb{E}(Z_1)$. The direct proof above avoids the regularity conditions needed to differentiate under the expectation. For the SPRT, $\mathbb{E}(N) < \infty$ by the almost-sure termination theorem, so Wald's equation applies provided $\mathbb{E}_\theta|Z_1| < \infty$ (true for every model in the section on distribution-specific SPRTs); it is the basis of the A.S.N. approximation.

Practice (Lecture Notes, Chapter 2)

- Short-answer questions (Section 2.12): Q8.
- Exercises (Section 2.13): 2.9.

Next lecture

Lecture 28: Wald's second identity.