

Lecture 28: Wald's second identity

Statistical Inference-II ■ Units II–III: Sequential Analysis

Kapil Kumar

Department of Statistics
Chaudhary Charan Singh University, Meerut

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Previous lecture

Lecture 27: The fundamental identity for the SPRT; Wald's equation.

Today

- Wald's Fundamental Identity and Its Uses
 - Wald's second identity

At a parameter value where $\mathbb{E}(Z_1) = 0$, both sides of Wald's equation vanish, so it gives no information about $\mathbb{E}(N)$. The following second identity is then needed.

Theorem 28.1 (Wald's second identity)

Let $Z_1, Z_2, \dots$ be i.i.d. with $\mathbb{E}(Z_1) = 0$ and $\sigma^2 = \mathbb{E}(Z_1^2) < \infty$, and let N be a stopping time with respect to a filtration $(\mathcal{F}_m)$ such that Z_m is $\mathcal{F}_m$ -measurable and Z_{m+1} is independent of $\mathcal{F}_m$ (e.g. $\mathcal{F}_m = \sigma(X_1, \dots, X_m)$ when Z_i is a function of X_i), with $\mathbb{E}(N) < \infty$. Then

$$\mathbb{E}(W_N^2) = \sigma^2 \mathbb{E}(N).$$

Proof.

Step 1: the truncated identity. For fixed n , $W_{N \wedge n} = \sum_{i=1}^n Z_i \mathbf{1}(N \geq i)$. Squaring,

$$\begin{aligned} \mathbb{E}(W_{N \wedge n}^2) &= \sum_{i=1}^n \mathbb{E}\{Z_i^2 \mathbf{1}(N \geq i)\} \\ &\quad + 2 \sum_{1 \leq i < j \leq n} \mathbb{E}\{Z_i \mathbf{1}(N \geq i) \mathbf{1}(N \geq j) Z_j\}. \end{aligned}$$

For $i < j$, the random variable $Z_i \mathbf{1}(N \geq i) \mathbf{1}(N \geq j)$ is $\mathcal{F}_{j-1}$ -measurable and hence independent of Z_j ; since $\mathbb{E}(Z_j) = 0$, every cross term vanishes. In the first sum Z_i is independent of $\{N \geq i\}$, so

$$\mathbb{E}(W_{N \wedge n}^2) = \sigma^2 \sum_{i=1}^n \mathbb{P}(N \geq i) = \sigma^2 \mathbb{E}(N \wedge n).$$

Proof (cont.)

Step 2: the limit. As $n \rightarrow \infty$, $\mathbb{E}(N \wedge n) \rightarrow \mathbb{E}(N)$ by monotone convergence, and it can be shown that $\mathbb{E}(W_{N \wedge n}^2) \rightarrow \mathbb{E}(W_N^2)$ (full proof: Lecture Notes, Appendix A). Hence $\mathbb{E}(W_N^2) = \sigma^2 \mathbb{E}(N)$. □

This identity is used in the subsection on the zero-drift case (section on the A.S.N. function) to approximate the A.S.N. at the zero-drift point.

Practice (Lecture Notes, Chapter 2)

- Exercises (Section 2.13): 2.17.

Next lecture

Lecture 29: The O.C. function: the root $h(\theta)$ and the O.C. formula.