

Lecture 29: The O.C. function: the root $h(\theta)$ and the O.C. formula

Statistical Inference–II ■ Units II–III: Sequential Analysis

Kapil Kumar

Department of Statistics
Chaudhary Charan Singh University, Meerut

M.Sc. Statistics, Semester III
Academic Session 2026–2027

This lecture

Previous lecture

Lecture 28: Wald's second identity.

Today

- Operating Characteristic Function

The operating characteristic (O.C.) function

The O.C. and A.S.N. functions are obtained from Wald's fundamental identity and Wald's equation (previous section).

The operating characteristic (O.C.) function is the probability of accepting H_0 when the true parameter is θ :

$$Q(\theta) = \mathbb{P}_\theta(\text{accept } H_0).$$

Thus $Q(\theta_0) = 1 - \alpha' \approx 1 - \alpha$ and $Q(\theta_1) = \beta' \approx \beta$, where α, β are the nominal and α', β' the attained error probabilities, and $1 - Q(\theta)$ is the power function of the SPRT. A desirable test has a large O.C. value in the region represented by H_0 and a small value in the region represented by H_1 .

Note 29.1 (Notation)

Wald (1947) and Goon, Gupta and Dasgupta (2003) write the O.C. function as $L(\theta)$. In these notes L denotes a loss function (Chapter 1), so the O.C. function is written $Q(\theta)$.

For a fixed θ , define the moment generating function of Z_1 ,

$$\phi_{\theta}(t) = \mathbb{E}_{\theta}(e^{tZ_1}) = \mathbb{E}_{\theta} \left[\left\{ \frac{f(X; \theta_1)}{f(X; \theta_0)} \right\}^t \right].$$

Lemma 29.2 (The O.C. root)

Suppose that $\mathbb{P}_{\theta}(Z_1 > 0) > 0$, $\mathbb{P}_{\theta}(Z_1 < 0) > 0$, $\mu_Z(\theta) = \mathbb{E}_{\theta}(Z_1) \neq 0$, and that $\phi_{\theta}(t)$ is finite for every real t . Then the equation

$$\phi_{\theta}(h) = 1$$

has exactly one non-zero root $h = h(\theta)$, and $h(\theta)$ has the sign opposite to that of $\mu_Z(\theta)$. Moreover,

$$h(\theta_0) = 1, \quad h(\theta_1) = -1.$$

Proof.

The function ϕ_θ is continuous and strictly convex (because Z_1 is not degenerate), with $\phi_\theta(0) = 1$ and $\phi'_\theta(0) = \mu_Z(\theta)$. Choose $c > 0$ such that $\mathbb{P}_\theta(Z_1 \leq -c) > 0$ and $\mathbb{P}_\theta(Z_1 \geq c) > 0$. For $t < 0$ we have $e^{tZ_1} \geq e^{-tc}$ on $\{Z_1 \leq -c\}$, so

$$\phi_\theta(t) \geq e^{-tc} \mathbb{P}_\theta(Z_1 \leq -c) \longrightarrow \infty \quad (t \rightarrow -\infty),$$

and similarly $\phi_\theta(t) \rightarrow \infty$ as $t \rightarrow +\infty$.

Suppose $\mu_Z(\theta) > 0$. Then $\phi_\theta(t) < 1$ for small negative t , while $\phi_\theta(t) \rightarrow \infty$ as $t \rightarrow -\infty$; by continuity there is a root $h < 0$. A strictly convex function takes the value 1 at most twice, and $t = 0$ is one of them, so there is no other non-zero root. The case $\mu_Z(\theta) < 0$ is symmetric.

Proof (cont.)

Finally,

$$\phi_{\theta_0}(1) = \mathbb{E}_{\theta_0} \left[\frac{f_1(X)}{f_0(X)} \right] = \int f_1(x) dx = 1,$$

$$\phi_{\theta_1}(-1) = \mathbb{E}_{\theta_1} \left[\frac{f_0(X)}{f_1(X)} \right] = \int f_0(x) dx = 1,$$

with sums in place of integrals in the discrete case. □

Remark 29.3

For the Bernoulli, binomial, Poisson and normal-mean models of the section on distribution-specific SPRTs, ϕ_θ is finite on the whole real line. For the normal-variance, exponential and gamma models it is finite only on an interval, but the conclusion of the lemma still holds (full proof: Lecture Notes, Appendix A).

Wald's fundamental identity for the SPRT (section on Wald's fundamental identity) with $t = h(\theta)$, for which $\phi_\theta\{h(\theta)\} = 1$, gives the *exact* relation

$$\mathbb{E}_\theta[\exp\{h(\theta)W_N\}] = 1.$$

Splitting this expectation according to the decision taken,

$$\begin{aligned} & Q(\theta) \mathbb{E}_\theta[e^{hW_N} \mid \text{accept } H_0] \\ & + \{1 - Q(\theta)\} \mathbb{E}_\theta[e^{hW_N} \mid \text{reject } H_0] = 1, \quad h = h(\theta). \end{aligned}$$

So far everything is exact.

The *only* approximation is to neglect the overshoot of W_N beyond the boundaries: $W_N \approx \log B$ when H_0 is accepted and $W_N \approx \log A$ when it is rejected, so the two conditional expectations are replaced by B^h and A^h . Consequently,

$$Q(\theta)B^{h(\theta)} + [1 - Q(\theta)]A^{h(\theta)} \approx 1,$$

and hence

$$Q(\theta) \approx \frac{A^{h(\theta)} - 1}{A^{h(\theta)} - B^{h(\theta)}}.$$

Practice (Lecture Notes, Chapter 2)

- Short-answer questions (Section 2.12): Q10.
- Exercises (Section 2.13): 2.10.

Next lecture

Lecture 30: The parametric method for drawing the O.C. curve.