

Lecture 31: The A.S.N. function

Statistical Inference–II ■ Units II–III: Sequential Analysis

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M.Sc. Statistics, Semester III
Academic Session 2026–2027

Previous lecture

Lecture 30: The parametric method for drawing the O.C. curve.

Today

- Average Sample Number Function

The average sample number (A.S.N.) function

The average sample number (A.S.N.) function is

$$\text{ASN}(\theta) = \mathbb{E}_\theta(N).$$

Write $\mu_Z(\theta) = \mathbb{E}_\theta(Z_1)$. Since $\mathbb{E}_\theta(N) < \infty$ (termination theorem), Wald's equation gives, provided $\mathbb{E}_\theta|Z_1| < \infty$ (true for every model in the section on distribution-specific SPRTs),

$$\mathbb{E}_\theta(W_N) = \mathbb{E}_\theta(N)\mu_Z(\theta).$$

Ignoring overshoot,

$$\mathbb{E}_\theta(W_N) \approx Q(\theta) \log B + [1 - Q(\theta)] \log A.$$

Therefore, provided $\mu_Z(\theta) \neq 0$,

$$\boxed{\mathbb{E}_\theta(N) \approx \frac{Q(\theta) \log B + [1 - Q(\theta)] \log A}{\mu_Z(\theta)}}.$$

The A.S.N. approximation is positive

Both numerator and denominator have the same sign. To see this, write $a = \log A > 0$, $b = \log B < 0$ and $h = h(\theta) \neq 0$, and substitute $Q = (A^h - 1)/(A^h - B^h)$. The numerator then satisfies

$$\begin{aligned}\{Qb + (1 - Q)a\}(A^h - B^h) &= (A^h - 1)b + (1 - B^h)a \\ &= g(h) := be^{ha} - ae^{hb} + a - b.\end{aligned}$$

Now $g(0) = 0$ and $g'(h) = ab(e^{ha} - e^{hb})$; since $ab < 0$ and $a > b$, $g'(h)$ has the sign of $-h$. Hence g increases on $(-\infty, 0]$ and decreases on $[0, \infty)$, so $g(h) < 0$ for every $h \neq 0$.

Since $A^h - B^h$ has the sign of h , the numerator $g(h)/(A^h - B^h)$ is negative exactly when $h(\theta) > 0$, which by the O.C. root lemma happens exactly when the drift $\mu_Z(\theta)$ is negative. Hence the approximation is positive.

In particular,

$$\mathbb{E}_{\theta_0}(N) \approx \frac{(1 - \alpha) \log B + \alpha \log A}{\mu_Z(\theta_0)},$$
$$\mathbb{E}_{\theta_1}(N) \approx \frac{\beta \log B + (1 - \beta) \log A}{\mu_Z(\theta_1)}.$$

Remark 31.1 (The drift under the two hypotheses)

$$\begin{aligned}\mu_Z(\theta_0) &= \mathbb{E}_{\theta_0} \log\{f_1(X)/f_0(X)\} = -K(f_0, f_1) < 0, \\ \mu_Z(\theta_1) &= K(f_1, f_0) > 0,\end{aligned}$$

where $K(f, g) = \mathbb{E}_f \log\{f(X)/g(X)\}$ is the Kullback–Leibler divergence, which is positive for $f \neq g$ by Jensen's inequality.

Under H_0 the walk W_m drifts downwards towards the acceptance boundary and under H_1 upwards towards the rejection boundary; the larger the divergence between f_0 and f_1 , the smaller the A.S.N.

Practice (Lecture Notes, Chapter 2)

- Short-answer questions (Section 2.12): Q9, Q14.
- Exercises (Section 2.13): 2.18.

Next lecture

Lecture 32: The zero-drift case.