

Lecture 35: SPRT for a normal mean and a normal variance

Statistical Inference–II ■ Units II–III: Sequential Analysis

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Previous lecture

Lecture 34: SPRT for a Poisson population.

Today

- Distribution-Specific SPRTs
 - Normal mean with known variance
 - Normal variance with known mean

Let $X_i \sim N(\mu, \sigma^2)$, where σ^2 is known, and test $H_0 : \mu = \mu_0$ against $H_1 : \mu = \mu_1, \mu_1 > \mu_0$. With $S_m = \sum_{i=1}^m X_i$,

$$Z_i = \frac{\mu_1 - \mu_0}{\sigma^2} \left(X_i - \frac{\mu_0 + \mu_1}{2} \right),$$
$$W_m = \frac{\mu_1 - \mu_0}{\sigma^2} \left(S_m - \frac{m(\mu_0 + \mu_1)}{2} \right).$$

Lines.

$$\frac{\sigma^2 \log B}{\mu_1 - \mu_0} + \frac{m(\mu_0 + \mu_1)}{2} < S_m < \frac{\sigma^2 \log A}{\mu_1 - \mu_0} + \frac{m(\mu_0 + \mu_1)}{2}.$$

Under μ , Z_1 is normal with

$$\mu_Z(\mu) = \frac{\mu_1 - \mu_0}{\sigma^2} \left(\mu - \frac{\mu_0 + \mu_1}{2} \right), \quad \sigma_Z^2 = \frac{(\mu_1 - \mu_0)^2}{\sigma^2}.$$

Hence $\phi_\mu(h) = \exp\{h\mu_Z + \frac{1}{2}h^2\sigma_Z^2\}$, and $\phi_\mu(h) = 1$ with $h \neq 0$ gives $h = -2\mu_Z/\sigma_Z^2$, that is,

$$h(\mu) = \frac{\mu_0 + \mu_1 - 2\mu}{\mu_1 - \mu_0}.$$

This is linear in μ , with $h(\mu_0) = 1$, $h(\mu_1) = -1$ and $h = 0$ at the midpoint $\mu^* = (\mu_0 + \mu_1)/2$.

Therefore

$$Q(\mu) \approx \frac{A^{h(\mu)} - 1}{A^{h(\mu)} - B^{h(\mu)}},$$

$$\mathbb{E}_{\mu}(N) \approx \frac{Q(\mu) \log B + \{1 - Q(\mu)\} \log A}{(\mu_1 - \mu_0) \{\mu - (\mu_0 + \mu_1)/2\} / \sigma^2}, \quad \mu \neq \mu^*,$$

and at the midpoint

$$\mathbb{E}_{\mu^*}(N) \approx -\frac{\sigma^2 (\log A) (\log B)}{(\mu_1 - \mu_0)^2}.$$

Let $X_i \sim N(\mu, \sigma^2)$ with known μ , and test $H_0 : \sigma^2 = \sigma_0^2$ against $H_1 : \sigma^2 = \sigma_1^2$, $\sigma_1 > \sigma_0$. Put

$$d = \frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2} > 0, \quad V_m = \sum_{i=1}^m (X_i - \mu)^2.$$

Then

$$Z_i = \frac{d}{2}(X_i - \mu)^2 - \log \frac{\sigma_1}{\sigma_0}, \quad W_m = \frac{d}{2}V_m - m \log \frac{\sigma_1}{\sigma_0}.$$

Lines. H_0 is accepted for small and rejected for large values of V_m :

$$\frac{2\{\log B + m \log(\sigma_1/\sigma_0)\}}{d} < V_m < \frac{2\{\log A + m \log(\sigma_1/\sigma_0)\}}{d}.$$

Under σ^2 , $(X - \mu)^2/\sigma^2 \sim \chi_1^2$, whose m.g.f. is $(1 - 2u)^{-1/2}$ for $u < \frac{1}{2}$.
Hence, for $h d \sigma^2 < 1$,

$$\phi_{\sigma^2}(h) = \left(\frac{\sigma_0}{\sigma_1}\right)^h (1 - h d \sigma^2)^{-1/2} = 1 \iff \left(\frac{\sigma_0}{\sigma_1}\right)^{2h} = 1 - h d \sigma^2,$$

so that

$$\sigma^2(h) = \frac{1 - (\sigma_0/\sigma_1)^{2h}}{h d}, \quad h \neq 0.$$

One verifies $\sigma^2(1) = \sigma_0^2$ and $\sigma^2(-1) = \sigma_1^2$.

$$\mu_Z(\sigma^2) = \frac{d}{2}\sigma^2 - \log \frac{\sigma_1}{\sigma_0}, \quad \sigma^{*2} = \frac{\log(\sigma_1^2/\sigma_0^2)}{d} = \frac{\log(\sigma_1^2/\sigma_0^2)}{\sigma_0^{-2} - \sigma_1^{-2}}.$$

Away from σ^{*2} ,

$$\mathbb{E}_{\sigma^2}(N) \approx \frac{Q(\sigma^2) \log B + [1 - Q(\sigma^2)] \log A}{\frac{1}{2}d\sigma^2 - \log(\sigma_1/\sigma_0)}.$$

At the zero-drift value, $\text{Var}\{(X - \mu)^2\} = 2\sigma^{*4}$ gives

$$\mathbb{E}_{\sigma^{*2}}(Z_1^2) = \frac{d^2}{4} \cdot 2\sigma^{*4} = \frac{d^2\sigma^{*4}}{2}, \quad \mathbb{E}_{\sigma^{*2}}(N) \approx -\frac{2(\log A)(\log B)}{d^2\sigma^{*4}}.$$

Practice (Lecture Notes, Chapter 2)

- Short-answer questions (Section 2.12): Q12.
- Exercises (Section 2.13): 2.5, 2.11.

Next lecture

Lecture 36: SPRT for exponential and gamma populations.