Lecture 36: SPRT for exponential and gamma populations

Statistical Inference–II ▪ Units II–III: Sequential Analysis

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This lecture

Previous lecture

Lecture 35: SPRT for a normal mean and a normal variance.

Today

- Distribution-Specific SPRTs
  - Exponential distribution: rate parameterisation
  - Gamma distribution with known shape

Exponential (rate): likelihood ratio

In this chapter the exponential parameter is a *rate*:

$$f(x;\theta) = \theta e^{-\theta x}, \qquad x > 0, \quad \theta > 0.$$

For testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$, $\theta_1 > \theta_0$ (shorter lifetimes under H_1), with $S_m = \sum_{i=1}^{m} X_i$,

$$\Lambda_m = \left(\frac{\theta_1}{\theta_0}\right)^m \exp\left\{-(\theta_1 - \theta_0)S_m\right\},$$

$$W_m = m \log \frac{\theta_1}{\theta_0} - (\theta_1 - \theta_0)S_m.$$

Exponential (rate): lines and O.C. function

Lines. Here W_m *decreases* in S_m, so H_0 is rejected for small and accepted for large total lifetimes:

$$\text{reject } H_0 \text{ if } S_m \le \frac{m\log(\theta_1/\theta_0) - \log A}{\theta_1 - \theta_0},$$

$$\text{accept } H_0 \text{ if } S_m \ge \frac{m\log(\theta_1/\theta_0) - \log B}{\theta_1 - \theta_0}.$$

O.C. function. Since $\mathbb{E}_\theta(e^{-uX}) = \theta/(\theta+u)$ for $\theta + u > 0$,

$$\phi_\theta(h) = \left(\frac{\theta_1}{\theta_0}\right)^h \frac{\theta}{\theta + h(\theta_1 - \theta_0)} = 1, \qquad \text{whenever } \theta + h(\theta_1 - \theta_0) > 0,$$

Exponential (rate): O.C. and A.S.N. functions

. . . which gives

$$\boxed{\theta(h) = \frac{h(\theta_1 - \theta_0)}{(\theta_1/\theta_0)^h - 1}, \qquad h \neq 0,}$$

with $\theta(1) = \theta_0$ and $\theta(-1) = \theta_1$. (Notice the formal identity with the Poisson case.)

A.S.N. function.

$$\mu_Z(\theta) = \log\frac{\theta_1}{\theta_0} - \frac{\theta_1 - \theta_0}{\theta}, \qquad \theta^* = \frac{\theta_1 - \theta_0}{\log(\theta_1/\theta_0)}.$$

Because $\mathrm{Var}_\theta(Z_1) = (\theta_1 - \theta_0)^2/\theta^2$,

$$\mathbb{E}_{\theta^*}(Z_1^2) = \left(\log\frac{\theta_1}{\theta_0}\right)^2, \qquad \mathbb{E}_{\theta^*}(N) \approx -\frac{(\log A)(\log B)}{\{\log(\theta_1/\theta_0)\}^2}.$$

Gamma (known shape): likelihood ratio

Let X_i have gamma density with known shape $k > 0$ and unknown rate $\theta > 0$:

$$f(x;\theta) = \frac{\theta^k}{\Gamma(k)} x^{k-1} e^{-\theta x}, \qquad x > 0.$$

For testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$, $\theta_1 > \theta_0$,

$$Z_i = k \log \frac{\theta_1}{\theta_0} - (\theta_1 - \theta_0) X_i, \qquad W_m = km \log \frac{\theta_1}{\theta_0} - (\theta_1 - \theta_0) S_m.$$

Gamma (known shape): acceptance and rejection lines

Lines.

$$\text{reject } H_0 \text{ if } S_m \le \frac{km\log(\theta_1/\theta_0) - \log A}{\theta_1 - \theta_0},$$

$$\text{accept } H_0 \text{ if } S_m \ge \frac{km\log(\theta_1/\theta_0) - \log B}{\theta_1 - \theta_0}.$$

Gamma (known shape): O.C. function

Since $\mathbb{E}_\theta(e^{-uX}) = \{\theta/(\theta+u)\}^k$,

$$\phi_\theta(h) = \left[\left(\frac{\theta_1}{\theta_0}\right)^h \frac{\theta}{\theta + h(\theta_1 - \theta_0)}\right]^k = 1$$

$$\iff \left(\frac{\theta_1}{\theta_0}\right)^h \frac{\theta}{\theta + h(\theta_1 - \theta_0)} = 1.$$

Thus the O.C. root, $\theta(h)$, and hence the O.C. curve are exactly those of the exponential case, for every shape k.

Gamma (known shape): A.S.N. function

$$\mu_Z(\theta) = k\left\{\log\frac{\theta_1}{\theta_0} - \frac{\theta_1-\theta_0}{\theta}\right\}, \qquad \theta^* = \frac{\theta_1-\theta_0}{\log(\theta_1/\theta_0)},$$

$$\mathbb{E}_{\theta^*}(Z_1^2) = k\left(\log\frac{\theta_1}{\theta_0}\right)^2, \qquad \mathbb{E}_{\theta^*}(N) \approx -\frac{(\log A)(\log B)}{k\{\log(\theta_1/\theta_0)\}^2}.$$

The drift is k times the exponential drift, so the A.S.N. is approximately the exponential A.S.N. divided by k. The exponential model is the special case $k = 1$. Stating explicitly whether θ is a rate or a scale is essential; the rate convention is used here for both gamma and exponential distributions.

Summary of common SPRTs: increments

Model	**Increment** Z_i	$\mathbb{E}_{\theta^*}(Z_1^2)$
Bernoulli	$X_i \log \frac{p_1}{p_0} + (1 - X_i) \log \frac{1-p_1}{1-p_0}$	$g_1 g_2$
Poisson	$X_i \log \frac{\lambda_1}{\lambda_0} - (\lambda_1 - \lambda_0)$	$\lambda^* \big(\log \frac{\lambda_1}{\lambda_0}\big)^2$
Normal mean	$\frac{\mu_1 - \mu_0}{\sigma^2} \big(X_i - \frac{\mu_0 + \mu_1}{2}\big)$	$\frac{(\mu_1 - \mu_0)^2}{\sigma^2}$
Normal variance	$\frac{d}{2}(X_i - \mu)^2 - \log \frac{\sigma_1}{\sigma_0}$	$\frac{1}{2} d^2 \sigma^{*4}$
Exponential rate	$\log \frac{\theta_1}{\theta_0} - (\theta_1 - \theta_0) X_i$	$\big(\log \frac{\theta_1}{\theta_0}\big)^2$
Gamma rate, shape k	$k \log \frac{\theta_1}{\theta_0} - (\theta_1 - \theta_0) X_i$	$k \big(\log \frac{\theta_1}{\theta_0}\big)^2$

Table 36.1: Log-likelihood-ratio increments, O.C. parametrisations (next frame) and zero-drift second moments for common SPRTs. In every case $\theta^* = \lim_{h \to 0} \theta(h)$ is the zero-drift value.

Summary of common SPRTs: O.C. parametrisations

Model	**O.C. parametrisation**
Bernoulli	$p(h) = \dfrac{1-\kappa^h}{\rho^h - \kappa^h}$, with $\rho = \frac{p_1}{p_0}$, $\kappa = \frac{1-p_1}{1-p_0}$
Poisson	$\lambda(h) = \dfrac{h(\lambda_1 - \lambda_0)}{(\lambda_1/\lambda_0)^h - 1}$
Normal mean	$\mu(h) = \frac{1}{2}\{\mu_0 + \mu_1 - h(\mu_1 - \mu_0)\}$
Normal variance	$\sigma^2(h) = \dfrac{1-(\sigma_0/\sigma_1)^{2h}}{hd}$
Exponential rate	$\theta(h) = \dfrac{h(\theta_1 - \theta_0)}{(\theta_1/\theta_0)^h - 1}$
Gamma rate, shape k	same as exponential

Table 36.1: (cont.) O.C. parametrisations.

After this lecture

Practice (Lecture Notes, Chapter 2)

- Short-answer questions (Section 2.12): Q13.
- Exercises (Section 2.13): 2.12.

Next lecture

Lecture 37: O.C. and A.S.N. curves; comparison with a fixed-sample test.