

Lecture 37: O.C. and A.S.N. curves; comparison with a fixed-sample test

Statistical Inference–II ■ Units II–III: Sequential Analysis

Kapil Kumar

Department of Statistics
Chaudhary Charan Singh University, Meerut

M.Sc. Statistics, Semester III
Academic Session 2026–2027

This lecture

Previous lecture

Lecture 36: SPRT for exponential and gamma populations.

Today

- O.C. and A.S.N. Curves: A Complete Worked Example

Example 37.1 (O.C. and A.S.N. of a Bernoulli SPRT)

A lot of items is to be judged acceptable if its fraction defective is $p_0 = 0.3$ and unacceptable if it is $p_1 = 0.5$. Items are inspected one at a time. Construct the SPRT with $\alpha = 0.05$ and $\beta = 0.10$, obtain its O.C. and A.S.N. functions, and compare the expected sample size with that of a fixed-sample test having the same error probabilities.

Solution

Boundaries. $A = 0.90/0.05 = 18$ and $B = 0.10/0.95 = 0.10526$, so $\log A = 2.8904$ and $\log B = -2.2513$.

Acceptance and rejection lines. Here $g_1 = \log(0.5/0.3) = 0.5108$, $g_2 = \log(0.7/0.5) = 0.3365$ and $g_1 + g_2 = 0.8473$. Hence

$$s = \frac{0.3365}{0.8473} = 0.3971,$$
$$\ell_0 = \frac{2.2513}{0.8473} = 2.6570, \quad \ell_1 = \frac{2.8904}{0.8473} = 3.4113,$$

Solution (cont.)

and after m items with T_m defectives:

accept the lot if $T_m \leq -2.657 + 0.3971 m$,

reject the lot if $T_m \geq 3.411 + 0.3971 m$,

and continue otherwise. For example, rejection is impossible before $m = 6$ (the rejection line exceeds m until then), and acceptance is impossible before $m = 7$ (the acceptance line is negative until then).

Solution (cont.)

O.C. and A.S.N. Using the parametric form of the Bernoulli SPRT,

$$p(h) = \frac{1 - (5/7)^h}{(5/3)^h - (5/7)^h}, \quad Q(h) = \frac{18^h - 1}{18^h - 0.10526^h},$$

$$\mathbb{E}_p(N) \approx \frac{Q \log B + (1 - Q) \log A}{0.8473 p - 0.3365},$$

with zero-drift value $p^* = 0.3971$. There

$$Q(p^*) \approx \frac{2.8904}{5.1417} = 0.5621, \quad \mathbb{E}_{p^*}(N) \approx \frac{(2.8904)(2.2513)}{(0.5108)(0.3365)} = 37.86.$$

Worked example: table of O.C. and A.S.N. values

Solution (cont.)

The resulting values are:

h	-3	-2	-1.5	-1	-0.5	0
$p(h)$	0.690	0.600	0.551	0.500	0.448	0.397
$Q(p)$	0.001	0.011	0.034	0.100	0.269	0.562
$\mathbb{E}_p(N)$	11.6	16.5	20.9	27.3	34.8	37.9

h	0.5	1	1.5	2	3
$p(h)$	0.347	0.300	0.256	0.216	0.149
$Q(p)$	0.828	0.950	0.987	0.997	1.000
$\mathbb{E}_p(N)$	32.3	24.2	18.3	14.6	10.7

Solution (cont.)

Fixed-sample comparison. A one-sided fixed-sample test based on the normal approximation needs

$$\begin{aligned} n &\approx \left\{ \frac{z_{0.05} \sqrt{p_0(1-p_0)} + z_{0.10} \sqrt{p_1(1-p_1)}}{p_1 - p_0} \right\}^2 \\ &= \left\{ \frac{1.6449(0.4583) + 1.2816(0.5)}{0.2} \right\}^2 = 48.6, \end{aligned}$$

that is, $n = 49$ items whatever the true p .

Worked example: fixed-sample comparison (cont.)

Solution (cont.)

The SPRT needs on average about 26 items when $p = p_0$ and 30 when $p = p_1$ (the exact values 26.35 and 29.89 of the remark on the accuracy of Wald's approximations; Wald's approximations give 24.2 and 27.3), savings of about 46% and 39%. Even its largest exact A.S.N., about 43 near $p \approx 0.40$, is still below 49.

(The value $n = 49$ rests on the normal approximation; an exact binomial calculation is given in the Lecture Notes, Appendix A.)

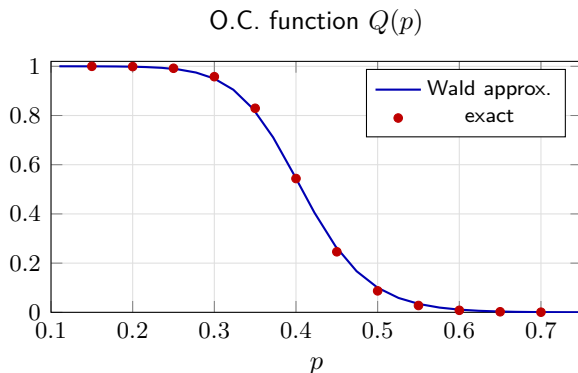


Figure 37.1: O.C. and A.S.N. functions of the SPRT of the worked example ($p_0 = 0.3$, $p_1 = 0.5$, $\alpha = 0.05$, $\beta = 0.10$). Curves: Wald's approximations. Points: exact values computed by propagating the probabilities of the integer states T_m stage by stage.

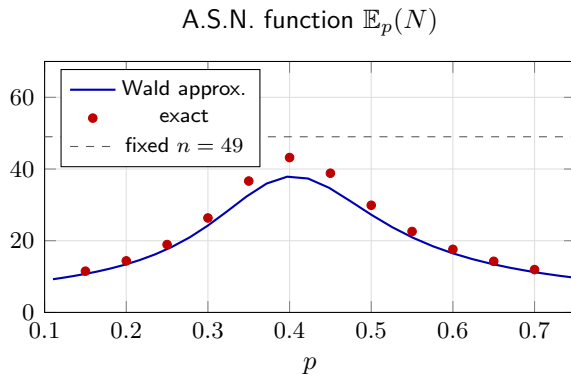


Figure 37.1: (cont.) A.S.N. panel. Curve: Wald's approximation; points: exact values; dashed line: the fixed-sample size $n = 49$.

How good are Wald's approximations?

Remark 37.2 (How good are Wald's approximations?)

Because T_m is integer-valued, the exact O.C. and A.S.N. of this test can be computed by propagating the probability distribution of T_m over the continuation states from one stage to the next. The exact values are:

p	$Q(p)$		$\mathbb{E}_p(N)$	
	Wald	exact	Wald	exact
0.3000 (p_0)	0.9500	0.9578	24.24	26.35
0.3971 (p^*)	0.5621	0.5630	37.86	43.12
0.5000 (p_1)	0.1000	0.0873	27.26	29.89

How good are Wald's approximations? (cont.)

Remark (cont.)

The attained error probabilities are $\alpha' = 1 - 0.9578 = 0.0422$ and $\beta' = 0.0873$, both below their nominal values, in agreement with Wald's bounds theorem ($\alpha' + \beta' = 0.1295 \leq 0.15$).

The A.S.N. approximation underestimates $\mathbb{E}_p(N)$, because neglecting the overshoot understates how far W_N travels. Both effects are typical.

Practice (Lecture Notes, Chapter 2)

- Exercises (Section 2.13): 2.14.

Next lecture

Lecture 38: Solved numerical examples: binomial, Poisson and normal.