Lecture 38: Solved numerical examples: binomial, Poisson and normal

Statistical Inference–II ▪ Units II–III: Sequential Analysis

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This lecture

Previous lecture

Lecture 37: O.C. and A.S.N. curves; comparison with a fixed-sample test.

Today

- Solved Numerical Examples

Common constants for the solved examples

All examples use $\alpha = 0.05$ and $\beta = 0.10$, so that

$$A = 18, \qquad B = 0.10526, \qquad \log A = 2.8904, \qquad \log B = -2.2513.$$

For the A.S.N. we also need

$$(1-\alpha)\log B + \alpha \log A = -1.9942,$$
$$\beta \log B + (1-\beta)\log A = 2.3762,$$
$$-(\log A)(\log B) = 6.5071.$$

Example: Bernoulli SPRT

Example 38.1 (Bernoulli SPRT)

Suppose X_i indicates whether the ith manufactured item is defective and $X_i \sim \text{Bernoulli}(p)$. Test

$$H_0 : p = 0.3 \qquad \text{against} \qquad H_1 : p = 0.5$$

with $\alpha = 0.05$ and $\beta = 0.10$. Apply the SPRT to the sample path $1, 1, 1, 1, 1, 1$.

Solution

The increment is

$$Z_i = X_i \log \frac{0.5}{0.3} + (1 - X_i) \log \frac{0.5}{0.7}.$$

Thus $Z_i = 0.5108$ when $X_i = 1$ and $Z_i = -0.3365$ when $X_i = 0$.

Example: Bernoulli SPRT (solution, cont.)

Solution (cont.)

For the given path,

m	1	2	3	4	5	6
W_m	0.5108	1.0217	1.5325	2.0433	2.5541	3.0650

The process remains between the boundaries through stage five, but $W_6 = 3.0650 > \log A$. The test stops at $N = 6$ and rejects H_0.

Equivalently, in terms of the lines of the complete worked example,

$$T_5 = 5 < 3.411 + 0.3971(5) = 5.397$$

$$\text{but} \quad T_6 = 6 \geq 3.411 + 0.3971(6) = 5.794.$$

Example: Poisson SPRT

Example 38.2 (Poisson SPRT)

Let X_i be the number of defects observed in hour i, with $X_i \sim \text{Poisson}(\lambda)$. Test

$$H_0 : \lambda = 2 \qquad \text{against} \qquad H_1 : \lambda = 4$$

with $\alpha = 0.05$, $\beta = 0.10$, using the observations

$$1, 3, 4, 2, 5, 3, 6, 4.$$

Also find the O.C. and A.S.N. at λ_0, λ_1 and the zero-drift value.

Example: Poisson SPRT (solution)

Solution

Here $\log(\lambda_1/\lambda_0) = \log 2 = 0.693147$ and

$$Z_i = X_i \log 2 - 2.$$

By the Poisson SPRT lines, sampling continues while

$$\frac{-2.2513 + 2m}{0.693147} < T_m < \frac{2.8904 + 2m}{0.693147},$$

$$\text{i.e.} \quad -3.248 + 2.885\, m < T_m < 4.170 + 2.885\, m.$$

Example: Poisson SPRT (solution, cont.)

Solution (cont.)

The cumulative values (computed with full precision and then rounded) are

m	1	2	3	4
T_m	1	4	8	10
rejection line	7.06	9.94	12.83	15.71
W_m	-1.3069	-1.2274	-0.4548	-1.0685

m	5	6	7	8
T_m	15	18	24	28
rejection line	18.60	21.48	24.37	27.25
W_m	0.3972	0.4766	2.6355	3.4081

Example: Poisson SPRT (solution, cont.)

Solution (cont.)

The acceptance line $(-0.36, 2.52, 5.41, \ldots, 19.84)$ is never reached. The first boundary crossing occurs at stage eight, where $T_8 = 28 \geq 27.25$, equivalently $W_8 = 3.4081 > \log A$. Hence the test rejects H_0 at $N = 8$.

O.C. and A.S.N. $Q(2) \approx 0.95$ and $Q(4) \approx 0.10$. The drifts are $\mu_Z(2) = 2\log 2 - 2 = -0.6137$ and $\mu_Z(4) = 4\log 2 - 2 = 0.7726$, so

$$\mathbb{E}_2(N) \approx \frac{-1.9942}{-0.6137} = 3.25, \qquad \mathbb{E}_4(N) \approx \frac{2.3762}{0.7726} = 3.08.$$

The zero-drift value is $\lambda^* = 2/\log 2 = 2.8854$, with $Q(\lambda^*) \approx 0.5621$ and $\mathbb{E}_{\lambda^*}(N) \approx 6.5071/\{2.8854(0.693147)^2\} = 4.69$.

Example: Normal mean SPRT

Example 38.3 (Normal mean SPRT)

The fill weight of packets is $N(\mu, 4)$. Test $H_0 : \mu = 10$ against $H_1 : \mu = 11$ with $\alpha = 0.05$, $\beta = 0.10$, using the observations

$$12.1,\ 10.8,\ 13.4,\ 11.9,\ 9.7,\ 12.6,\ 13.0,\ 11.5,\ 12.8.$$

Also obtain the O.C. and A.S.N. functions and compare with the fixed-sample test.

Solution

Here $\mu_1 - \mu_0 = 1$, $\sigma^2 = 4$ and $(\mu_0 + \mu_1)/2 = 10.5$. By the normal-mean SPRT lines, sampling continues while

$$4(-2.2513) + 10.5m < S_m < 4(2.8904) + 10.5m,$$

$$\text{i.e.} \quad -9.005 + 10.5\,m < S_m < 11.562 + 10.5\,m.$$

Example: Normal mean SPRT (solution, cont.)

Solution (cont.)

m	1	2	3	4	5
S_m	12.1	22.9	36.3	48.2	57.9
lower line	1.50	12.00	22.50	33.00	43.50
upper line	22.06	32.56	43.06	53.56	64.06
W_m	0.400	0.475	1.200	1.550	1.350

m	6	7	8	9
S_m	70.5	83.5	95.0	107.8
lower line	54.00	64.50	75.00	85.50
upper line	74.56	85.06	95.56	106.06
W_m	1.875	2.500	2.750	3.325

Since $S_8 = 95.0 < 95.56$ but $S_9 = 107.8 \geq 106.06$, the test stops at $N = 9$ and rejects H_0.

Example: Normal mean SPRT (solution, cont.)

Solution (cont.)

O.C. and A.S.N. Here $h(\mu) = (21 - 2\mu)/1 = 21 - 2\mu$ and $\mu_Z(\mu) = (\mu - 10.5)/4$. Hence

μ	9.5	10	10.25	10.5	10.75	11	11.5
h	2	1	0.5	0	-0.5	-1	-2
$Q(\mu)$	0.997	0.950	0.828	0.562	0.269	0.100	0.011
$\mathbb{E}_\mu(N)$	8.94	15.95	21.84	26.03	24.16	19.01	11.33

where, for instance, $\mathbb{E}_{10}(N) \approx -1.9942/(-0.125) = 15.95$ and, at the midpoint, $\mathbb{E}_{10.5}(N) \approx 4(6.5071)/1 = 26.03$.

Example: Normal mean SPRT (solution, cont.)

Solution (cont.)

A fixed-sample one-sided z-test with the same α and β needs

$$n = \{(z_{0.05} + z_{0.10})\sigma/(\mu_1 - \mu_0)\}^2 = \{(2.9265)(2)\}^2 = 34.3,$$

i.e. 35 observations, more than the SPRT needs on average even near $\mu = 10.5$, where its A.S.N. is largest.

(Wald's A.S.N. curve peaks at about 26.16 at $\mu \approx 10.55$. Wald's approximation underestimates the exact A.S.N.; see Lecture Notes, Appendix A.)

After this lecture

Practice (Lecture Notes, Chapter 2)

- Exercises (Section 2.13): 2.15.

Next lecture

Lecture 39: Solved numerical examples: exponential and gamma.