Lecture 40: Summary and revision

Statistical Inference–II ▪ Units II–III: Sequential Analysis

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This lecture

Previous lecture

Lecture 39: Solved numerical examples: exponential and gamma.

Today

- Summary and Formula Sheet
- Short-Answer Questions

Key ideas of Units II–III

1. In a sequential test the sample size N is a random stopping time. The SPRT continues while the log-likelihood ratio W_m lies between $\log B$ and $\log A$.
2. W_m is a random walk; under H_0 it drifts down, under H_1 it drifts up. The SPRT terminates with probability one, and all moments of N are finite.
3. Wald's boundaries $A \approx (1-\beta)/\alpha$, $B \approx \beta/(1-\alpha)$ are safe: the attained error probabilities can exceed the nominal ones only slightly, and their sum cannot exceed $\alpha + \beta$.

Key ideas (cont.)

Key ideas of Units II–III (cont.)

4. Wald's fundamental identity gives the O.C. function; Wald's equation gives the A.S.N. function; Wald's second identity handles the zero-drift point. The only approximation is neglect of the overshoot.
5. For one-parameter exponential families the SPRT reduces to two parallel straight lines for a running total S_m.
6. The SPRT typically needs far fewer observations on average than the fixed-sample test with the same error probabilities.

Formula sheet

Quantity	**Formula**
Increments and walk	$Z_i = \log\{f(X_i;\theta_1)/f(X_i;\theta_0)\}$, $W_m = \sum_{i=1}^{m} Z_i = \log \Lambda_m$
SPRT	accept H_0 if $W_m \le \log B$; reject if $W_m \ge \log A$; continue otherwise
Boundaries	$A \approx \dfrac{1-\beta}{\alpha}, \qquad B \approx \dfrac{\beta}{1-\alpha}$
Wald's bounds	$\alpha' \le \dfrac{\alpha}{1-\beta}, \quad \beta' \le \dfrac{\beta}{1-\alpha}, \quad \alpha' + \beta' \le \alpha + \beta$
Termination	$\mathbb{P}_\theta(N > n) \le \rho^{\lfloor n/k \rfloor}$, $0 < \rho < 1$; $\quad \mathbb{E}_\theta(N^r) < \infty$

Formula sheet (cont.)

Quantity	**Formula**
Fundamental identity	$\mathbb{E}\left[e^{tW_N}\phi(t)^{-N}\right] = 1, \quad \phi(t) = \mathbb{E}(e^{tZ_1})$ (N bounded; or SPRT with $\phi(t) \geq 1$)
Wald's equation	$\mathbb{E}(W_N) = \mathbb{E}(N)\,\mathbb{E}(Z_1)$
Second identity	$\mathbb{E}(W_N^2) = \mathbb{E}(N)\,\mathbb{E}(Z_1^2)$ when $\mathbb{E}(Z_1) = 0$
O.C. root	$\phi_\theta(h) = 1,\ h \neq 0; \quad h(\theta_0) = 1,\ h(\theta_1) = -1$
O.C. function	$Q(\theta) \approx \dfrac{A^h - 1}{A^h - B^h}$; at $h = 0$: $Q \approx \dfrac{\log A}{\log A - \log B}$

Formula sheet (cont.)

Quantity	**Formula**
A.S.N. function	$\mathbb{E}_\theta(N) \approx \dfrac{Q\log B + (1-Q)\log A}{\mu_Z(\theta)}$, $\mu_Z(\theta) = \mathbb{E}_\theta(Z_1) \neq 0$
Zero drift	$\mathbb{E}_{\theta^*}(N) \approx -\dfrac{(\log A)(\log B)}{\mathbb{E}_{\theta^*}(Z_1^2)}$
Bernoulli lines	continue while $-\ell_0 + s\,m < T_m < \ell_1 + s\,m$, $s = \dfrac{g_2}{g_1+g_2}$, $\ell_0 = \dfrac{-\log B}{g_1+g_2}$, $\ell_1 = \dfrac{\log A}{g_1+g_2}$
Other models	see the summary tables of common SPRTs (section on distribution-specific SPRTs)

Short-answer questions (Q1–Q7)

Q1. What is a sequential test? How does it differ from a fixed-sample test?

Q2. Define a stopping time. Why must the decision to stop at stage m depend only on $X_1, \ldots, X_m$?

Q3. Define Wald's SPRT for testing a simple hypothesis against a simple alternative.

Q4. Write Wald's approximations to the boundaries A and B in terms of α and β.

Q5. State Wald's bounds on the attained error probabilities.

Q6. Why are the attained error probabilities of an SPRT usually smaller than the nominal ones?

Q7. State the theorem on termination of the SPRT with probability one.

Short-answer questions (Q8–Q14)

Q8. State Wald's fundamental identity and Wald's equation.

Q9. Define the O.C. and A.S.N. functions of an SPRT.

Q10. What are the values of $h(\theta)$ at θ_0 and θ_1, and what are $Q(\theta_0)$ and $Q(\theta_1)$?

Q11. Why can the usual A.S.N. formula not be used at the zero-drift point? What replaces it?

Q12. Show that the SPRT for a normal mean depends on the data only through $\sum X_i$.

Q13. Why does the O.C. curve of the gamma SPRT not depend on the shape parameter?

Q14. Interpret the sign of $\mathbb{E}_{\theta_0}(Z_1) = -K(f_0, f_1)$.

After this lecture

Practice (Lecture Notes, Chapter 2)

- Short-answer questions (Section 2.12): Q1–Q14 (all, for revision).

Next lecture

Lecture 41 begins Units IV–V: Nonparametric Inference.