Lecture 43: Order statistics; the distribution of $X_{(r)}$

Statistical Inference–II ▪ Units IV–V: Nonparametric Inference

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This lecture

Previous lecture

Lecture 42: Empirical distribution function; probability integral transformation.

Today

- Order Statistics and Estimation of Quantiles
  - Distribution of an Order Statistic

Order statistics and population quantiles

Let

$$X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$$

denote the order statistics from a random sample.

Definition 43.1 (Population quantile)

For $0 < p < 1$, the population pth quantile is

$$\xi_p = F^{-1}(p) = \inf\{x : F(x) \ge p\}.$$

The median is $\xi_{1/2}$, the lower quartile is $\xi_{1/4}$, and the upper quartile is $\xi_{3/4}$. Throughout this chapter the population median is also denoted by $\eta = \xi_{1/2}$.

Sample quantiles

A natural sample quantile is obtained from the order statistic near the index np. One simple definition is

$$\widehat{\xi}_p = X_{(\lceil np \rceil)}.$$

Software packages use several interpolation conventions; the convention must therefore be reported when numerical answers are compared.

Distribution of an order statistic

For any distribution function F, continuous or not, the distribution function of $X_{(r)}$ is

$$\mathbb{P}\{X_{(r)} \le x\} = \sum_{j=r}^{n} \binom{n}{j} F(x)^j \{1 - F(x)\}^{n-j},$$

because $X_{(r)} \le x$ exactly when at least r observations are $\le x$, and the number of such observations is $\mathrm{Bin}\{n, F(x)\}$.

If F has density f, the density of $X_{(r)}$ is

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} F(x)^{r-1} \{1 - F(x)\}^{n-r} f(x).$$

Distribution of $F\{X_{(r)}\}$

If F is continuous, then, since F is non-decreasing,

$$F\{X_{(r)}\} = U_{(r)}$$

is the rth order statistic of $U_i = F(X_i)$, which are i.i.d. $U(0,1)$ by the probability integral transformation.

Applying the density formula with $F(u) = u$ and $f(u) = 1$ on $(0,1)$ gives

$$F\{X_{(r)}\} = U_{(r)} \sim \text{Beta}(r, n-r+1).$$

After this lecture

Practice (Lecture Notes, Chapter 3)

- Short-answer questions (Section 3.14): Q3.
- Exercises (Section 3.15): 3.4.

Next lecture

Lecture 44: Sample quantiles: consistency and asymptotic normality.