

Lecture 54: Wilcoxon signed-rank test: null distribution

Statistical Inference-II ■ Units IV–V: Nonparametric Inference

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This lecture

Previous lecture

Lecture 53: Median test for two samples; Fisher's exact test.

Today

- Wilcoxon Signed-Rank Test
 - One-Sample Formulation

The Wilcoxon signed-rank test

The signed-rank test uses both signs and ranks of absolute deviations. It is more informative than the sign test, but its standard location interpretation requires symmetry.

Suppose $X_1, \dots, X_n$ are independent from a continuous distribution that is symmetric about θ . To test $H_0 : \theta = \theta_0$, form $D_i = X_i - \theta_0$, delete zero differences, rank the nonzero $|D_i|$ from smallest to largest, and attach the sign of D_i to each rank. From now on n denotes the number of nonzero differences (the n^* of the sign test). Let

W^+ = sum of positive ranks, W^- = sum of negative ranks.

Since

$$W^+ + W^- = \frac{n(n+1)}{2},$$

either statistic determines the other. A common two-sided statistic is

$$T = \min(W^+, W^-).$$

Theorem 54.1 (Null distribution of the signed-rank statistic)

Suppose that, under H_0 , $D_1, \dots, D_n$ are i.i.d. from a continuous distribution symmetric about 0. For $k = 1, \dots, n$ let $V_k = 1$ if the observation whose absolute value has rank k is positive, and $V_k = 0$ otherwise. Then

$$W^+ = \sum_{k=1}^n k V_k,$$

where $V_1, \dots, V_n$ are i.i.d. Bernoulli($\frac{1}{2}$). Consequently:

- (i) $\mathbb{P}(W^+ = w) = c_n(w)/2^n$, where $c_n(w)$ is the number of subsets of $\{1, \dots, n\}$ whose elements sum to w ; equivalently

$$\sum_w \mathbb{P}(W^+ = w) t^w = \frac{1}{2^n} \prod_{k=1}^n (1 + t^k),$$

$$c_n(w) = c_{n-1}(w) + c_{n-1}(w - n);$$

Theorem (Null distribution of the signed-rank statistic, cont.)

- (ii) the distribution of W^+ is symmetric about $n(n+1)/4$;
- (iii) when there are no ties,

$$\mathbb{E}(W^+) = \frac{n(n+1)}{4}, \quad \text{Var}(W^+) = \frac{n(n+1)(2n+1)}{24}.$$

Proof of Theorem: Null distribution of the signed-rank statistic

Proof.

For a continuous distribution symmetric about 0, $|D|$ and the sign of D are independent and $\mathbb{P}(D > 0) = \mathbb{P}(D < 0) = \frac{1}{2}$: indeed, for every $x > 0$,

$$\begin{aligned}\mathbb{P}(D > 0, |D| \leq x) &= \mathbb{P}(0 < D \leq x) \\ &= \mathbb{P}(-x \leq D < 0) = \mathbb{P}(D < 0, |D| \leq x),\end{aligned}$$

so each equals $\frac{1}{2}\mathbb{P}(|D| \leq x)$.

Because the pairs $(|D_i|, \text{sign } D_i)$ are independent across i , the vector of signs is independent of the vector of absolute values, and hence of the ranks of the absolute values.

Therefore, whatever the ranking, the signs attached to ranks $1, \dots, n$ are i.i.d. equally likely, which proves the representation.

Proof (cont.)

(i) Each of the 2^n sign patterns has probability 2^{-n} , and $W^+ = w$ exactly when the set of ranks carrying a positive sign sums to w . The generating function follows from independence, since $\mathbb{E}(t^{kV_k}) = (1 + t^k)/2$. The recursion classifies the subsets according to whether or not they contain n .

(ii) Replacing every V_k by $1 - V_k$ leaves the joint distribution unchanged and maps W^+ to $n(n+1)/2 - W^+ = W^-$; hence W^+ and $n(n+1)/2 - W^+$ have the same distribution.

(iii) $\mathbb{E}(V_k) = \frac{1}{2}$ and $\text{Var}(V_k) = \frac{1}{4}$, so by independence

$$\mathbb{E}(W^+) = \frac{1}{2} \sum_{k=1}^n k = \frac{n(n+1)}{4},$$

$$\text{Var}(W^+) = \frac{1}{4} \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{24}.$$



The large-sample statistic is

$$Z = \frac{W^+ - n(n+1)/4}{\sqrt{n(n+1)(2n+1)/24}},$$

with a continuity correction. Exact critical values for $n \leq 30$: Tables in Appendix B of the lecture notes. For a one-sided test use the rank sum that is small under H_1 ; for a two-sided test use $T = \min(W^+, W^-)$.

If the absolute differences contain tie groups of sizes $t_1, t_2, \dots$ and average ranks are assigned, then

$$\text{Var}(W^+) = \frac{n(n+1)(2n+1)}{24} - \frac{1}{48} \sum_j t_j(t_j^2 - 1).$$

Wilcoxon signed-rank test

1. Compute deviations from the hypothesised centre, or paired differences.
2. Delete zero differences and record the effective sample size.
3. Rank absolute differences, assigning average ranks to ties.
4. Sum positive and negative ranks.
5. Use the exact distribution for a small effective sample or the corrected normal approximation for a large sample.

Example 54.2

Apply the signed-rank test to the differences

$$-1, 2, 3, 4, 5, 6.$$

Solution

The absolute values already have ranks 1, 2, 3, 4, 5, 6. Thus

$$W^- = 1, \quad W^+ = 2 + 3 + 4 + 5 + 6 = 20, \quad T = 1.$$

There are $2^6 = 64$ equally likely sign assignments under the null. Only the subsets $\emptyset$ and $\{1\}$ of $\{1, \dots, 6\}$ have sum at most 1, so

$\mathbb{P}(W^- \leq 1) = 2/64$, and by symmetry the exact two-sided probability of a value at least this extreme is $4/64 = 0.0625$. The result is suggestive but does not reject at the 5% level.

Practice (Lecture Notes, Chapter 3)

- Exercises (Section 3.15): 3.17, 3.18.

Next lecture

Lecture 55: Paired signed-rank test with ties; Hodges–Lehmann estimator.