

Lecture 58: One-sample Kolmogorov–Smirnov test

Statistical Inference–II ■ Units IV–V: Nonparametric Inference

Kapil Kumar

Department of Statistics
Chaudhary Charan Singh University, Meerut

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Previous lecture

Lecture 57: Mann–Whitney test with ties; normal approximation; Hodges–Lehmann shift estimator.

Today

- Kolmogorov–Smirnov Tests
 - One-Sample K–S Test
 - One-Sided K–S Tests

Kolmogorov–Smirnov (K–S) procedures compare empirical distribution functions and are sensitive to general distributional discrepancies; they focus on the largest vertical distance between distribution functions.

Unlike the sign and rank statistics, the one-sample K–S statistic uses the magnitudes of the observations, through the values $F_0(X_{(i)})$.

The two-sample statistic, like the rank-sum statistic, depends on the data only through the pooled ordering of the two samples.

Let F_0 be a completely specified continuous distribution function and let $H_0 : F = F_0$. The one-sided and two-sided K-S statistics are defined by

$$D_n^+ = \sup_x \{\hat{F}_n(x) - F_0(x)\}, \quad D_n^- = \sup_x \{F_0(x) - \hat{F}_n(x)\},$$

$$D_n = \sup_x |\hat{F}_n(x) - F_0(x)| = \max(D_n^+, D_n^-).$$

Because $\hat{F}_n$ is constant between consecutive order statistics and F_0 is continuous and non-decreasing, the suprema can be computed from the ordered observations $X_{(1)} < \cdots < X_{(n)}$:

$$D_n^+ = \max_{1 \leq i \leq n} \left\{ \frac{i}{n} - F_0(X_{(i)}) \right\}, \quad D_n^- = \max_{1 \leq i \leq n} \left\{ F_0(X_{(i)}) - \frac{i-1}{n} \right\}.$$

These formulas hold for continuous F_0 ; their derivation is Exercise 3.25.

Under H_0 , $U_{(i)} = F_0(X_{(i)})$ are the order statistics of a $U(0, 1)$ sample (probability integral transformation), and the formulas express D_n^+ , D_n^- and D_n through $U_{(1)}, \dots, U_{(n)}$ alone. Hence the null distribution of each statistic is the same for every continuous, fully specified F_0 . As $n \rightarrow \infty$,

$$\mathbb{P}(\sqrt{n}D_n \leq z) \longrightarrow K(z) = 1 - 2 \sum_{k=1}^{\infty} (-1)^{k-1} e^{-2k^2 z^2}, \quad z > 0.$$

A common large-sample rule rejects when $D_n > c_\alpha / \sqrt{n}$, where

α	0.10	0.05	0.025	0.01
c_α	1.22	1.36	1.48	1.63

are standard asymptotic constants, $K(c_\alpha) = 1 - \alpha$.

The exact null distribution of D_n is continuous. Its upper α -points $d_{n,\alpha}$, defined by

$$\mathbb{P}(D_n > d_{n,\alpha}) = \alpha \quad \text{under } H_0,$$

are tabulated on the next frame for $n \leq 20$, and H_0 is rejected at level α when $D_n > d_{n,\alpha}$.

For $n > 20$ the asymptotic value $c_\alpha/\sqrt{n}$ is commonly used (some tables switch only for $n > 35$). It slightly overstates the exact value, so the large-sample test is mildly conservative: for $\alpha = 0.05$,

- $1.36/\sqrt{20} = 0.304$ against the exact 0.294;
- $1.36/\sqrt{35} = 0.230$ against the exact 0.224.

Exact critical values $d_{n,\alpha}$ of D_n

n	$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.01$	n	$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.01$
1	0.950	0.975	0.995	11	0.352	0.391	0.468
2	0.776	0.842	0.929	12	0.338	0.375	0.449
3	0.636	0.708	0.829	13	0.325	0.361	0.432
4	0.565	0.624	0.734	14	0.314	0.349	0.418
5	0.509	0.563	0.669	15	0.304	0.338	0.404
6	0.468	0.519	0.617	16	0.295	0.327	0.392
7	0.436	0.483	0.576	17	0.286	0.318	0.381
8	0.410	0.454	0.542	18	0.279	0.309	0.371
9	0.387	0.430	0.513	19	0.271	0.301	0.361
10	0.369	0.409	0.489	20	0.265	0.294	0.352

$n > 20$: approximately $1.22/\sqrt{n}$, $1.36/\sqrt{n}$ and $1.63/\sqrt{n}$, respectively.

Two-sided one-sample K-S test: $\mathbb{P}(D_n > d_{n,\alpha}) = \alpha$ under H_0 (computed from the exact null distribution of D_n).

Example: testing uniformity

Example 58.1 (Testing uniformity)

For the ordered sample

0.10, 0.25, 0.40, 0.70, 0.90

test $H_0 : F(x) = x$, $0 < x < 1$ (uniformity on $(0, 1)$), at the 5% level.

Solution

Here $F_0(X_{(i)}) = X_{(i)}$ and $n = 5$. By the computational formulas,

i	1	2	3	4	5
$X_{(i)}$	0.10	0.25	0.40	0.70	0.90
$i/5 - X_{(i)}$	0.10	0.15	0.20	0.10	0.10
$X_{(i)} - (i - 1)/5$	0.10	0.05	0.00	0.10	0.10

so that $D_5^+ = 0.20$, $D_5^- = 0.10$ and $D_5 = 0.20$.

Example: testing uniformity (cont.)

Solution (cont.)

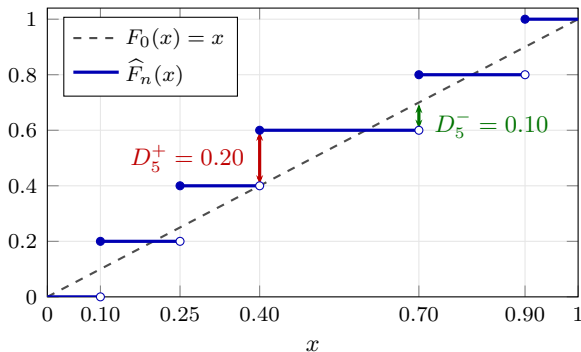
D_5^+ is attained at $x = 0.40$, where $\hat{F}_n$ jumps to 0.6, and D_5^- is approached just before the jumps at 0.10, 0.70 and 0.90 (figure on the next frame).

From the table of exact critical values, $d_{5,0.05} = 0.563$. Since $D_5 = 0.20 < 0.563$, the null is not rejected; indeed the exact p -value is

$$\mathbb{P}(D_5 \geq 0.20) = 0.962.$$

The large-sample value $1.36/\sqrt{5} \approx 0.608$ overstates the exact critical value; for such a small sample the exact value should be used.

The e.d.f. and F_0 for the uniformity example



Steps: $\hat{F}_n$ (filled dots mark the value at each jump); dashed: $F_0(x) = x$. D_5^+ occurs at $x = 0.40$; D_5^- is approached just before the jump at $x = 0.70$ (and also at 0.10 and 0.90).

Remark 58.2 (Estimated parameters)

The ordinary one-sample K–S critical values assume that F_0 is completely specified before observing the data. If parameters such as a normal mean and variance are estimated from the same sample, the null distribution changes. A Lilliefors-type correction or a parametric bootstrap is then required.

D_n^+ is large when the empirical distribution tends to lie above F_0 , whereas D_n^- is large in the opposite direction.

The direction must be translated carefully into stochastic ordering: a distribution function lying above another corresponds to observations tending to be smaller, since $\mathbb{P}(X > x) = 1 - F(x)$. Accordingly, to test $H_0 : F = F_0$:

- against $H_1^+ : F(x) \geq F_0(x)$ for all x , with strict inequality for some x (X stochastically *smaller* than under F_0), reject for large D_n^+ ;
- against $H_1^- : F(x) \leq F_0(x)$ for all x , with strict inequality for some x (X stochastically *larger* than under F_0), reject for large D_n^- .

Theorem 58.3 (Null distribution of the one-sided statistics)

Let F_0 be continuous and suppose $H_0 : F = F_0$ holds. Then

- (i) D_n^+ and D_n^- have the same distribution, which does not depend on F_0 ;
- (ii) for every $z \geq 0$, $\mathbb{P}(\sqrt{n} D_n^+ \leq z) \rightarrow 1 - e^{-2z^2}$ as $n \rightarrow \infty$, and the same limit holds for D_n^- .

Proof.

(i) By the computational formulas, $D_n^+ = \max_i \{i/n - U_{(i)}\}$ and $D_n^- = \max_i \{U_{(i)} - (i-1)/n\}$, where $U_{(1)} < \dots < U_{(n)}$ are the order statistics of the i.i.d. $U(0, 1)$ variables $U_i = F_0(X_i)$; so neither null distribution depends on F_0 . Put $V_i = 1 - U_i$: again i.i.d. $U(0, 1)$, with $V_{(i)} = 1 - U_{(n+1-i)}$. Writing $k = n + 1 - i$,

$$\max_i \left\{ \frac{i}{n} - V_{(i)} \right\} = \max_i \left\{ U_{(n+1-i)} - \frac{n-i}{n} \right\} = \max_k \left\{ U_{(k)} - \frac{k-1}{n} \right\} = D_n^-.$$

The left side is D_n^+ computed from the uniform sample $V_1, \dots, V_n$, which has the same distribution as D_n^+ computed from $U_1, \dots, U_n$. Hence D_n^- and D_n^+ have the same distribution.

(ii) This limit is due to Smirnov (1939); its proof is beyond this course. By (i) it applies equally to D_n^- . □

Consequently an asymptotic level- α test rejects when $D_n^+ > c_\alpha^+/\sqrt{n}$ (respectively $D_n^- > c_\alpha^+/\sqrt{n}$), where $1 - e^{-2(c_\alpha^+)^2} = 1 - \alpha$, that is

$$c_\alpha^+ = \sqrt{-(\ln \alpha)/2} : \quad \begin{array}{c|cccc} \alpha & 0.10 & 0.05 & 0.025 & 0.01 \\ \hline c_\alpha^+ & 1.073 & 1.224 & 1.358 & 1.517 \end{array}$$

Note that $c_{\alpha/2}^+ \approx c_\alpha$ (for instance $c_{0.025}^+ = 1.358$ and $c_{0.05} = 1.36$), because the two-sided tail

$$1 - K(z) = 2e^{-2z^2} - 2e^{-8z^2} + \dots$$

is very nearly twice the one-sided tail e^{-2z^2} in this range of z .

For small n , exact one-sided tail probabilities can be computed (Lecture Notes, Appendix A).

For the data of the uniformity example, $D_5^+ = 0.20$ gives

$$\mathbb{P}(D_5^+ \geq 0.20) = 0.585 :$$

there is no evidence that the observations are stochastically smaller than uniform.

Practice (Lecture Notes, Chapter 3)

- Short-answer questions (Section 3.14): Q13, Q14.
- Exercises (Section 3.15): 3.24, 3.25, 3.27.

Next lecture

Lecture 59: Two-sample Kolmogorov–Smirnov test.