

Lecture 60: Comparison and selection of procedures; summary and revision

Statistical Inference–II ■ Units IV–V: Nonparametric Inference

Kapil Kumar

Department of Statistics
Chaudhary Charan Singh University, Meerut

M.Sc. Statistics, Semester III
Academic Session 2026–2027

Previous lecture

Lecture 59: Two-sample Kolmogorov–Smirnov test.

Today

- Comparison and Selection of Procedures
- Summary and Formula Sheet
- Short-Answer Questions

Comparison of procedures (1/2)

Procedure	Design	Primary null feature	Important assumptions
Runs test	Ordered one-sample or pooled two-sample labels	Random order or equality of distributions	Independence/exchangeability; continuity for pooled ranks
Sign test	One sample or matched pairs	Median or median paired difference	Independence and continuity; no symmetry requirement
Median test	Two or more independent samples	Homogeneity relative to pooled median	Independent samples; common-shape interpretation for a pure median conclusion
Wilcoxon signed-rank	One sample or matched pairs	Symmetric location centre	Independence, continuity and symmetry of deviations

Comparison of procedures (2/2)

Procedure	Design	Primary null feature	Important assumptions
Mann–Whitney rank-sum	Two independent samples	Equality of distributions; location shift under common shape	Independence, continuity; common shape for location interpretation
One-sample K–S	One sample	Fully specified distribution F_0	Continuous and completely specified F_0
Two-sample K–S	Two independent samples	Equality of entire distributions	Independence and continuity

A practical selection rule

Use the sampling design first, not the observed p -values, to choose a test.

For paired observations use a paired procedure; for independent groups use an independent-sample procedure.

Next identify the target: median, symmetric location, stochastic ordering, equality of complete distributions or randomness.

Finally check continuity, symmetry, ties and sample size before choosing exact or asymptotic calibration.

Key ideas of Units IV–V

1. Distribution-free tests get their null distributions from symmetry, random labelling, signs, ranks or the probability integral transformation, not from a parametric model.
2. If F is continuous, $F(X) \sim U(0, 1)$ and $F(X_{(r)}) \sim \text{Beta}(r, n - r + 1)$; quantile intervals come from $\text{Bin}(n, p)$ probabilities.
3. Choose the test from the design (one sample, paired, two or k independent samples) and the target (median, symmetric centre, location shift, whole distribution, randomness).
4. Sign test: no symmetry needed. Signed-rank test: needs symmetry, more powerful. Mann–Whitney: location shift under a common shape. K–S: whole distribution.
5. Use exact tables for small samples; otherwise the normal approximation with continuity correction and tie correction.

Procedure	Null distribution and large-sample form
E.d.f.	$\mathbb{E}\hat{F}_n(x) = F(x), \quad \text{Var } \hat{F}_n(x) = F(x)\{1 - F(x)\}/n$
Sample quantile	$\sqrt{n}(\hat{\xi}_p - \xi_p) \rightarrow N(0, p(1-p)/f(\xi_p)^2)$
Quantile CI	$\mathbb{P}\{X_{(r)} \leq \xi_p \leq X_{(s)}\} = \sum_{j=r}^{s-1} \binom{n}{j} p^j (1-p)^{n-j}$
Runs	$\mathbb{E}(R) = 1 + \frac{2n_1 n_2}{n}, \quad \text{Var}(R) = \frac{2n_1 n_2 (2n_1 n_2 - n)}{n^2 (n-1)}$
Runs up and down	$\mathbb{E}(R_{ud}) = \frac{2n-1}{3}, \quad \text{Var}(R_{ud}) = \frac{16n-29}{90}$

Procedure	Null distribution and large-sample form
Sign test	$S_+ \sim \text{Bin}(n^*, \frac{1}{2}); \quad Z = \frac{S_+ - n^*/2}{\sqrt{n^*/4}}$
Median test	$X^2 = \sum (O_{ij} - E_{ij})^2 / E_{ij} \approx \chi_{k-1}^2, \quad E_{ij} = O_{i+} O_{+j} / N;$ 2 × 2: Fisher exact
Signed-rank	$\mathbb{E}(W^+) = \frac{n(n+1)}{4},$ $\text{Var}(W^+) = \frac{n(n+1)(2n+1)}{24} - \frac{1}{48} \sum t_j(t_j^2 - 1)$
Mann-Whitney	$U_X = R_X - \frac{m(m+1)}{2}, \quad U_X + U_Y = mn;$ $\mathbb{E}(U_X) = \frac{mn}{2},$ $\text{Var}(U_X) = \frac{mn}{12} \left[N + 1 - \frac{\sum (t_j^3 - t_j)}{N(N-1)} \right]$

Procedure	Null distribution and large-sample form
One-sample K-S	$D_n^+ = \max_i \{i/n - F_0(X_{(i)})\},$ $D_n^- = \max_i \{F_0(X_{(i)}) - (i-1)/n\};$ <p>reject if $D_n > c_\alpha / \sqrt{n}$, $c_{0.05} = 1.36$ (exact values for $n \leq 20$)</p>
Two-sample K-S	$D_{m,n} = \sup_x F_m(x) - G_n(x) ; \quad \text{reject if}$ $D_{m,n} > c_\alpha \sqrt{(m+n)/(mn)}$
Continuity correction	lower tail: $T + \frac{1}{2}$; upper tail: $T - \frac{1}{2}$

Short-answer questions (Q1–Q5)

- Q1. Distinguish between a nonparametric and a distribution-free procedure.
- Q2. State the probability integral transformation. Why is continuity of F needed?
- Q3. Write the distribution of $F(X_{(r)})$ for a sample from a continuous distribution.
- Q4. How is a distribution-free confidence interval for the median obtained from order statistics?
- Q5. Define a run. What do too few runs and too many runs indicate?

Short-answer questions (Q6–Q10)

- Q6. Write the mean and variance of the number of runs.
- Q7. State the null distribution of the sign-test statistic.
- Q8. How are zero differences and ties handled in the sign and signed-rank tests?
- Q9. Why does the signed-rank test require symmetry, while the sign test does not?
- Q10. Define the Mann–Whitney statistic and state the relation between U_X and the rank sum R_X .

Short-answer questions (Q11–Q15)

- Q11. Give the mean and variance of U_X under H_0 .
- Q12. What does a significant median test show?
- Q13. Define the one-sample Kolmogorov–Smirnov statistic.
- Q14. Why are the usual K–S critical values invalid when parameters of F_0 are estimated from the data?
- Q15. What is a continuity correction? Give its direction for a lower-tail probability.

Practice (Lecture Notes, Chapter 3)

- Short-answer questions (Section 3.14): Q1–Q15 (all, for revision).
- Exercises (Section 3.15): 3.28, 3.29, 3.30.

End of the course

This is the last lecture of Statistical Inference–II.